Individual Sections of the Book

Inverse Problems: Exercices

With *mathematica*, *matlab*, and *scilab* solutions

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March 12, 2007

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3.5 Fissures and Rifting

3.5.1 Problem

The inspiration for this example comes from my work with measuring and interpreting the deformation produced by episodes of rifting in Afar (Tarantola et al., 1979). When a fissure (dyke) opens in a rift valley (because magma produces “hydraulic fracturing”), the surrounding region relaxes elastic stresses, and, therefore, undergoes elastic deformations. The position, length, orientation, and opening of the fissure, may not be known, because the erupted lava may hide the fissure or because the fissure may be under water (we had both situations in Afar). Using a mechanical model of the Earth crust, and given the position, the length, the orientation, and the value of the opening of the fissure (these are the “model parameters”), one can predict the displacements produced in the region when a fissure opens (this is the “forward problem”). Let us here try to solve the “inverse problem” of estimating the parameters of the fissure, given the observed displacements at a set of geodetic points.

As the use of a realistic three-dimensional, elastic, modelisation code would require more time than that available for this exercise, I will use a dramatically simplified model of deformation (where elastic theory, in fact, is not used at all).

The Mechanical Model (Solving the Forward Problem)

Consider a two-dimensional plane, representing the surface of the (flat) Earth. A horizontal fissure is represented by the coordinates \((X, Y)\) of its center, by its length \(\Delta\), the angle \(\psi\) corresponding to the azimuth of a vector normal to the fissure, and by the value \(Q\) of the opening at the center of the fissure (see figure 3.11). As the parameters \(\Delta\) and \(Q\) are positive, and the statistics of positive parameters are tricky, we rather use the logarithmic parameters

\[
\delta = \log \Delta \quad ; \quad q = \log Q .
\] (3.1)

Therefore, we have

\[
\Delta = e^\delta \quad ; \quad Q = e^q .
\] (3.2)

Let \((x, y)\) be the coordinates of the point at which we wish to evaluate the displacements produced by the opening of the fissure (see figure 3.11), and let \(u\) be the (horizontal) displacement vector of this point \((x, y)\). Let us simply model this displacement \(u\) by assuming that it is

- proportional to the opening of the fissure, \(Q = e^q\);
- proportional to

\[
f_1 = \exp(-U/\Delta) ,
\] (3.3)

where \(U\) is the distance from point \((x, y)\) to the center of the fissure, and \(\Delta\) is the fissure length; this means that displacements decay exponentially away from the fissure, with the fissure length \(\Delta\) as the parameter of the exponential decay (an assuredly crude “mechanical” model);

- proportional to

\[
f_2 = \sin^2 \alpha ,
\] (3.4)
where $\alpha$ is the azimuth of the point $(x, y)$ as seen from the center of the fissure (see figure 3.11); this, again, is a rather crude way of taking into account the directionality effects;

- is colinear to the vector $\mathbf{U}$ going from the center of the fissure to the point $(x, y)$ (and has the same sense).

Therefore, for the (horizontal) displacement vector at point $(x, y)$, we take the expression

$$u = Q f_1 f_2 \frac{\mathbf{U}}{||\mathbf{U}||} ,$$

i.e.,

$$u = e^q \exp(-U/\Delta) \sin^2 \alpha \frac{\mathbf{U}}{U} .$$

This has two (horizontal) components, $u_x$, and $u_y$, the norm of this horizontal vector being $u_H = \sqrt{u_x^2 + u_y^2}$. In addition to this horizontal displacement, there may be a vertical one, that we (simplistically, just for this numerical exercise) assume to equal one-seventh of the horizontal displacement:

$$u_z = u_H/7 .$$

I leave as an exercise to the student to write the intermediary equations expressing $\Delta$, $\alpha$, and $\mathbf{U}$ as a function of the basic parameters $\{X, Y, \delta, \psi, q\}$ (these equations are written using Mathematica code in figure 3.11).

Figure 3.11: The variables and equations of the problem.
### 3.5.2 Solution

Assume that the following displacements have been observed:

- At point \((x_1, y_1) = (11.0, 24.5)\) : 
  \[ u_x = +0.016 \pm 0.01, \quad u_y = +0.099 \pm 0.01, \quad u_z = +0.022 \pm 0.01 \]
- At point \((x_2, y_2) = (13.0, 28.5)\) : 
  \[ u_x = +0.016 \pm 0.01, \quad u_y = +0.015 \pm 0.01, \quad u_z = +0.000 \pm 0.01 \]
- At point \((x_3, y_3) = (14.0, 21.5)\) : 
  \[ u_x = +0.075 \pm 0.01, \quad u_y = +0.028 \pm 0.01, \quad u_z = +0.006 \pm 0.01 \]
- At point \((x_4, y_4) = (11.0, 24.5)\) : 
  \[ u_x = +0.003 \pm 0.01, \quad u_y = +0.006 \pm 0.01, \quad u_z = -0.002 \pm 0.01 \]
- At point \((x_5, y_5) = (13.0, 28.5)\) : 
  \[ u_x = -0.077 \pm 0.01, \quad u_y = -0.083 \pm 0.01, \quad u_z = +0.018 \pm 0.01 \]
- At point \((x_6, y_6) = (14.0, 21.5)\) : 
  \[ u_x = -0.020 \pm 0.01, \quad u_y = +0.032 \pm 0.01, \quad u_z = +0.009 \pm 0.01 \]

This data set is represented in figure 3.12.

We have only six displacement vectors, and they have quite large uncertainties, so we can expect that our parameters are not going to be well resolved by the data.

The exercise we will solve together (in the computer room) will consist in using these data to estimate the five parameters \(\{X, Y, \delta, \psi, q\}\) that represent the fissure. We will first use very weak a priori information on the parameters, then, we will play the game of inputting a priori information, and/or of decreasing the number or quality of the data.

There are three Mathematica codes ready, one using the least-squares theory, another code using a slightly more general formulation, and a third code using the general probabilistic formulation.
Figure 3.13: The data and a few sample points of the a posteriori probability distribution in the model space. These samples convey an explicit information on the actual information we have on the position and size of the fissure. See an enlargement in figure 3.14.
3.5 Fissures and Rifting

You may choose to learn how to use *Mathematica* (you can have a look at the “ten-minute tutorial” that presents itself at start up of the program), or you may choose to write your own code, using some other mathematical software (like *matlab*).

Using the philosophy presented in this course, the non-unicity of the solution is displayed by showing some samples of the “a posteriori probability distribution in the model space”. The samples obtained using the least-squares formulation are displayed in figures 3.13 and 3.14.

![Figure 3.14: An enlargement of the fissures displayed in figure 3.13. A strong anti-correlation is visible between the length of the fissure and its opening.](image)

The usual way of presenting the results of a least-squares solution correponds to the following results:

\[
X = 10.1 \pm 0.2 \quad ; \quad Y = 19.8 \pm 0.3 \quad ; \quad \delta = 1.0 \pm 0.3 \quad ; \quad \psi = 0.5 \pm 0.1 \quad ; \quad q = -0.4 \pm 0.5 .
\]

(3.9)

The posterior correlation matrix is:

\[
R = \begin{pmatrix}
1 & -0.28 & -0.04 & -0.55 & +0.10 \\
-0.28 & 1 & -0.10 & +0.65 & +0.05 \\
-0.04 & -0.10 & 1 & +0.21 & -0.99 \\
-0.55 & +0.65 & +0.21 & 1 & -0.28 \\
+0.10 & +0.05 & -0.99 & -0.28 & 1
\end{pmatrix} .
\]

(3.10)

We see, indeed, that the parameters \( \delta \) and \( q \) are strongly anti-correlated.

**Bibliography**


3.5.3 Mathematica Notebook

Note: I have not yet had time to clean the code and to insert it in the text. For the time being, please use the following links.

Executable notebooks at

http://www.ipgp.jussieu.fr/~tarantola/exercises/chapter_03/AfarProbability.nb
http://www.ipgp.jussieu.fr/~tarantola/exercises/chapter_03/AfarFindMinimum.nb
http://www.ipgp.jussieu.fr/~tarantola/exercises/chapter_03/AfarLeastSquares.nb