Individual Sections of the Book

Inverse Problems: Exercises

With mathematica, matlab, and scilab solutions

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2.7 Period of a Pendulum [PROVISIONAL TEXT]

2.7.1 Question

We are told that the period $T$ of a periodical phenomenon is of the order of 33 seconds, with a relative uncertainty of the order of 25%. Choose an arbitrary probability density (mathematically consistent) that may represent this ‘a priori information’.

A complicated instrument is built to indirectly measure this period $T$. For any actual value of $T$, the instrument produces a point $(x, y)$ on an Euclidean plane, the Cartesian coordinates $(x, y)$ depending on $T$ as follows:

$$x = \alpha T^2 \quad ; \quad y = \beta T$$

(2.64)

with $\alpha = 1.0 \text{ m/s}^2$ and $\beta = 3.0 \text{ m/s}$. An actual measurement has provided the values

$$x = 2500 \text{ m} \pm 600 \text{ m} \quad ; \quad y = 150 \text{ m} \pm 60 \text{ m}$$

(2.65)

where it is assumed that the uncertainties are of the Gaussian type, and independent. Use the theory of inverse problems to obtain a ‘posterior’ probability density for $T$ that combines this experimental result with the a priori information.

Indicate what would change in the computations if the uncertainties in the measurement of $x$ and $y$ were not independent, but had a correlation $\rho = -0.9$.

All documents are allowed. Don’t rewrite the course, just solve this special problem, motivating the choices made.

2.7.2 Solution

As a period $T$ is a Jeffreys parameter, the simplest probability model to represent our a priori information is the lognormal probability density

$$\bar{\rho}_m(T) = \frac{1}{\sqrt{2\pi} \sigma} \frac{1}{T} \exp \left( -\frac{1}{2\sigma^2} \left( \log \frac{T}{T_0} \right)^2 \right).$$

(2.66)

A little bit of thinking should convince you that the special information we have (the period is of the order of 33 seconds, with a relative uncertainty of the order of 25%) corresponds to the values

$$T_0 = 33 \text{ s} \quad ; \quad \sigma = 0.25$$

(2.67)

This gives the probability density represented in figure 2.12.

Because the parameters $x$ and $y$ are Cartesian coordinates, and adequate way of representing the result of the measurements is by using a normal model. Assuming independent uncertainties,

$$\bar{\rho}_d(x, y) = \frac{1}{\sqrt{2\pi} \sigma_1} \exp \left( -\frac{(x-x_{\text{obs}})^2}{2\sigma_1^2} \right) \frac{1}{\sqrt{2\pi} \sigma_2} \exp \left( -\frac{(y-y_{\text{obs}})^2}{2\sigma_2^2} \right),$$

(2.68)

where

$$x_{\text{obs}} = 2500. \quad ; \quad y_{\text{obs}} = 150. \quad ; \quad \sigma_1 = 600. \quad ; \quad \sigma_2 = 60.$$
Figure 2.12: The probability density that represents the a priori information on the period.

Figure 2.13: The probability density for the observable parameters.

This probability density is represented in figure 2.13.

The posterior probability density is obtained using the fundamental equation of the course,

\[ \sigma_m(m) = \frac{1}{\nu} \rho_m(m) \rho_d(g(m)) , \]  

(2.70)

where \( \nu \) is a normalizing constant. Here, this gives, explicitly,

\[ \sigma_m(T) = \frac{1}{\nu} \rho_m(T) \rho_d(\alpha T^2, \beta T) , \]  

(2.71)

where the values \( \alpha \) and \( \beta \) are given in the text of the problem.

If we replace here the expressions for \( \rho_m(T) \) and for \( \rho_d(x, y) \), we obtain

\[ \sigma_m(T) = \frac{1}{\nu} \frac{1}{T} \exp \left( - \frac{1}{2\sigma_2^2} \left( \log \frac{T}{T_0} \right)^2 \right) \exp \left( - \frac{(\alpha T^2 - x_{obs})^2}{2\sigma_1^2} \right) \exp \left( - \frac{(\beta T - y_{obs})^2}{2\sigma_2^2} \right) , \]  

(2.72)

where, this time, the normalization constant \( \nu \) is to be obtained by numerical integration. This probability density is represented in figure 2.14.

WARNING: I still have to add here the case where \( \rho = -0.9 \).

### 2.7.3 Mathematica Code

Executable notebook at
[http://www.ipgp.jussieu.fr/~tarantola/exercices/chapter_02/PeriodOfPendulum.nb](http://www.ipgp.jussieu.fr/~tarantola/exercices/chapter_02/PeriodOfPendulum.nb)
Figure 2.14: The probability density that represents the a posteriori information on the period. To be compared with figure 2.12.

Note: I have not yet had time to insert these lines of code into the text, and to comment them. For the time being, here is a raw copy of the lines.

```math
LogNormalDensity[T_, T0_, sigma_] := 1/(Sqrt[2Pi] sigma) (1/T) Exp[-1/(2 sigma^2) Log[T/T0]^2]

rhoM[T_] := LogNormalDensity[T, 33, 0.25]
Plot[rhoM[T], {T, 0, 100}, Frame -> True, PlotRange -> {{-5, 105}, {-0.005, 0.080}}, GridLines -> {{0}, {0}}]

rhoD[x_, y_] := 1/(Sqrt[2Pi] sigmax) Exp[-(x - xobs)^2/(2 sigmax^2)]/(
             Sqrt[2Pi] sigmay) Exp[-(y - yobs)^2/(2 sigmay^2)]
xobs = 2500.; yobs = 150.; sigmax = 600.; sigmay = 60.;
DensityPlot[-rhoD[x, y], {x, 700, 4300}, {y, -30, 330}, Mesh -> False, PlotRange -> All, PlotPoints -> 51]

sigmaM[T_] := rhoM[T] rhoD[alpha T^2, beta T]
alpha = 1.0; beta = 3.0;
sigmaMNormaZ[rhoM[T] rhoD[alpha T^2, beta T]] := (1/nu) sigmaM[T]
INIntegrate[sigmaMNormaZ[T], {T, 0, Infinity}]
Plot[sigmaMNormaZ[T], {T, 0, 110}, Frame -> True, PlotRange -> {{-5, 105}, {-0.005, 0.080}}, GridLines -> {{0}, {0}}]
```