Logarithmic parameters

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Abstract

Some physical parameters, that are positive by definition, appear often as reciprocal pairs, like an electrical resistivity ρ and the electrical conductivity $\sigma = 1/\rho$, or a frequency ν and the period $T = 1/\nu$. Theoretical arguments suggest that behind these pairs there are more natural parameters: their logarithms. This paper suggests a terminology that could bring those logarithmic units into normal use. Although perturbing long-term established habitudes, the defined logarithmic units, if adopted, could simplify measuring practices: if the instruments were manufactured using these units, changes of scales (say from Ω to $\mathrm{k}\Omega$ and to $\mathrm{M}\Omega$ when measuring an electric resistence) would be suppressed, and the simpler algebra of absolute uncertainties would replace that of relative uncertainties in most measurements. The suppression of the arbitrary accumulation near the zero value when not using the logaritmic units (for instance, near the frequency zero in one case or near the period zero in the reciprocal case), may be important conceptually. As an example of use of the logarithmic units one may consider the tuning of a musical instrument where a new, slightly modified musical scale can be defined. The oddly defined *decibel* unit can be clarified using the proposed definitions.

1 Introduction

There is plenty of parameters that are positive by definition and that appear as pairs of reciprocal parameters. Some examples are:

- An electrical resistivity ρ and the electrical conductivity $\sigma = 1/\rho$;
- A frequency ν and the period $T = 1/\nu$;
- A rate of exchange of money, like when using the USD/FF ratio in France or the reciprocal FF/USD ratio in the US.

These parameters have been often analyzed, in particular with respect to information theory. For instance, Jeffreys (1) and Jaynes (2) argue that the state of total ignorance about the possible value of such a parameter can not be described by a constant probability density. The argument goes as follows.

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Consider, for instance, an electric resistivity ρ , or the associated electric conductivity $\sigma = 1/\rho$. To any probability density $f(\rho)$ over ρ will be associated an equivalent probability density $g(\sigma)$ over σ . Well known rules of change of variables give

$$g(\sigma) = \left| \frac{d\rho}{d\sigma} \right| f(\rho) = \frac{1}{\sigma^2} f(\rho) = \frac{1}{\sigma^2} f(\frac{1}{\sigma}) \ . \tag{1}$$

The probability density $f(\rho) = \text{const.}$ can not describe the state of total ignorance on the possible values of an electric conductivity, as it would imply, using equation 1, the probability density $g(\sigma) = \text{const.}/\sigma^2$ over the possible values of the electric conductivity. The probability densities $f(\rho)$ and $g(\sigma)$ would then have different form. But, to describe the state of total ignorance, any argument that can be applied to ρ can be applied to σ . Then, the two functions $f(\cdot)$ and $g(\cdot)$ have to be, in fact, identical. This constraint, toguether with the constraint of equation 1 leads, up to a multiplicative constant, to the unique solution

$$f(\rho) = \frac{1}{\rho} \qquad ; \qquad g(\sigma) = \frac{1}{\sigma} \ . \tag{2}$$

This (unnormalizable) probability density, does not allow to compute the absolute probability of an interval, which is natural if one considers that we do not have any information on the possible values of the parameter.

The interpretation of equations 2 is better grasped if introducing logarithmic parameters. Defining, for instance,

$$\rho^* = \log \frac{\rho}{\rho_0} \qquad ; \qquad \sigma^* = \log \frac{\sigma}{\sigma_0} \ , \tag{3}$$

where ρ_0 and σ_0 are arbitrary constants, and using the well known rules of change of variables leads to the new probability densities

$$f^*(\rho^*) = \text{const.} \qquad ; \qquad g^*(\sigma^*) = \text{const.} \tag{4}$$

This is telling us that although perfectly valid, the usual parameters resistivity-conductivity, satisfying $\sigma = 1/\rho$, although perfectly valid, have a nontrivial behaviour, as the total ignorance is not described by a non-constant probability density. Their logarithmic counterparts, $\rho^\star$ and $\sigma^\star$, satisfying $\sigma^\star = -\rho^\star$, are simple parameters, for which the total ignorance is described by constant probability densities. In some sense, the parameters ρ and σ have a "nonlinear behaviour", which is "ironed out" when taking their logarithms.

Although not formalized, taking the logarithm of the kinds of parameters explored in this paper is universal. Here are some examples.

1. Figures displaying a spectrum of a phenomenon (amplitude as a function of frequency) use a "logarithmic axis" for the frequencies. This
   (a) gives larger "dynamics" to the display and
   (b) avoids collapsing information near the zero frequency when not using a logaritmic axis.

2. Descriptions of processes near the begining of a Big Bang model of the Universe use an exponential notation ($10^{-27}\,\mathrm{s}$ after the Big Bang...). This is disguised way of using a "logarithmic axis".

3. The same is true when describing low temperatures: in the racecourse to the low temperature record, we talk about $10^{-3}\,\mathrm{K}$, $10^{-4}\,\mathrm{K}$ ...

It is my feeling that the importance of the logarithmic parameters has not been emphasized enough. Although, most often, they are effectively introduced, they are not recognized has having an independent existence. From where some common errors, as the one of intuitively assuming that a constant probability density may describe the total ignorance on a parameter like a frequency.

Only in engineering, though the introduction of the *decibel* measuring system, an effective introduction of logarithmic parameters is made. But its definition is odd, with factors of two between different uses of the decibel concept. And, in any case, the decibel scale is not aimed at measuring physical parameters, but rather to compare relative attenuations, amplifications, etc.

The next section defines logarithmic parameters and suggests a uniform nomenclature. The following sections explain some of the advantages of the explicit use of the new parameters, both conceptual and practical, and give a couple of examples.

2 Definition of logarithmic units

2.1 The example frequency-period

Let us start with a particular example, and give later the general definition. Consider that we have to measure the frequency ν of some phenomenon, or, correspondingly, its period $T = 1/\nu$.

The unit of frequency is the *hertz* (we will later recall its definition). This unit is realised, for instance, by any pendulum having as period $T = 1s$. The electric power supplied at homes realizes the frequency of $50\,\mathrm{hz}$ or $60\,\mathrm{hz}$, depending on the countries.

The *logarithmic frequency* associated to a frequency ν must have the form

$$\nu^* = a\,\log_b \frac{\nu}{\nu_0} \ , \tag{5}$$

where ν_0 is some fixed frequency. Obviously, the simplest definition will correspond to the choice of natural logarithms (base $b = e = 2.71828\ldots$) and to the choice $a = 1$. Then,

$$\nu^* = \log \frac{\nu}{\nu_0} \ , \tag{6}$$

where "log" without any index means the natural logarithm (we will mention below the use of different logarithm basis).

It remains to decide about the constant ν_0 . That such a constant is necessary to the definition is clear if we consider that the logarithm can only accept as argument a pure number, without physical dimensions.

Logarithmic scales (as the decibel scale) are typically used for relative measurements. For instance, when measuring the logarithmic amplitude of a sound one usually compares the pressure amplitude of a sound with the amplitude of a "normal conversation pressure amplitude" (taken as 10^{-1} Pa). But is there any fundamental difference between a relative and an absolute measurement? After all, an "absolute measurement" of a pressure is also a comparison of a pressure value to a reference unit, the pascal or the bar.

Then, the logarithmic parameters defined here through an expression like 6 are intended for a general use, absolute or relative measurements. In an absolute measurement, as one compares the measured value to the unit (for instance, a hertz, a second, a pascal or an ohm), the constant ν_0 will be that unit (respectively 1 hz, 1 s, 1 Pa or 1 Ω). In a relative measurement, one will compare the measured value to some reference value, different from the legal unit (for instance, a "normal conversation pressure amplitude" $p_0 = 10^{-1}\,\mathrm{Pa}$, the "musical middle A pitch", $\nu_0 = 440\,\mathrm{hz}$). The terminology to be defined below will make clear which is the reference chosen for measurement.

What about the units for a logarithmic frequency? By definition, ν^* , being the logarithm of a pure number, is itself a pure number, having no physical dimensions. But there is no reason for not giving a name to the unit of logarithmic frequency: an illustrious example of physical unit with a name but without physical dimensions is the *radian*, used to measure angles, and defined as the ratio between two lengths.

When making an absolute measure of a logarithmic frequency, using equation 6, we take $\nu_0 = 1\,\mathrm{hz}$. I propose to call *neperhertz* the unit of (absolute) logarithmic frequency, as reference to the natural, or Neperian, logarithms used in its definition. Then, for instance, to the frequency $\nu = 1.23\,10^4\,\mathrm{hz}$ we can now associate the logarithmic frequency $\nu^* = 9.42$ neperhertz .

Note: mention somewhere that the International Standards Organization (ISO) considers a decaying wave of form $F(t) = A\exp(-\alpha t)\cos(\omega t)$ and, it gives to ωt the name *radian* (rad), it gives to αt the name *neper* (Np). Then, while ω has physical dimensions rad/s , α has dimensions Np/s . [ISO 31-2:1992].

Turning to the period $T = 1/\nu$, we can introduce the logarithmic period

$$T^* = \log \frac{T}{T_0} \ . \tag{7}$$

In an absolute measurement, T_0 will be the unit of time duration $T_0 = 1\,\mathrm{s}$. Calling *nepersecond* the unit of logarithm time, we can associate to the period $T = 1/\nu = 8.13\,10^{-5}\,s$, the logarithmic period $T^\star = -9.42$ *nepersecond* .

Two obvious remarks. First, when choosing for the constant ν_0 the hertz, we have the simple rules

$$\nu = \alpha \text{ hertz} \ \Leftrightarrow\ \nu^* = (\log \alpha) \text{ neperhertz} \quad ; \quad \nu^\star = \beta \text{ neperhertz} \ \Leftrightarrow\ \nu = e^\beta \text{ hetz} \tag{8}$$

and, when choosing for the constant T_0 the second,

$$T = \gamma \text{ second} \ \Leftrightarrow\ T^* = (\log \gamma) \text{ nepersecond} \quad ; \quad T^\star = \delta \text{ nepersecond} \ \Leftrightarrow\ T = e^\delta \text{ second} \ . \tag{9}$$

Second, if the period T is the reciprocal of the period ν (i.e., if $T = 1/\nu$), then, $T^* = -\nu^*$, i.e., using the adimensional units,

$$T^\star = \varepsilon \text{ nepersecond} \ \Leftrightarrow\ \nu^\star = -\varepsilon \text{ neperhertz} \ . \tag{10}$$

Let us now turn to the terminology for relative measurements. When measuring, for instance, a logarithmic pressure

$$p^\star = \log \frac{p}{p_0} \ , \tag{11}$$

I have proposed, when $p_0 = 1\,\text{pascal}$, to use for $p^\star$ the unit neperpascal. It is adimensional and, in fact, it is just the number 1 (as the radian when measuring angles). When taking for p_0 the "normal conversation pressure amplitude" (ncpa) $p_0 = 10^{-1}\,\text{Pa}$, to perform a relative measurement, I propose to use for $p^\star$ the unit "neper-ncpa" (in complete words, a "neper-normal-conversation-pressure-amplitude").

Passing from this to the general case is obvious. In section 2.5 the link will be made with the decibel scale of engineers.

Note: should I give here more properties for the relative logarithmic measurements?

2.2 General case

The principle just suggested can be extended to all other parameters of the type here considered (pairs of positive, reciprocal parameters, whose probability density representing the state of total ignorance is of the form $1/x$). If a physical parameter has the value r , and if r_0 is the unit of this physical parameter, then, the logarithmic parameter is

$$r^* = \log \frac{r}{r_0} \tag{12}$$

and, for whatever the unit may be, if we say

$$r = \alpha \ \text{unit} \ , \tag{13}$$

then, we also say

$$r^* = (\log \alpha) \ \text{neperunit} \ , \tag{14}$$

or, reciprocally,

$$r^* = \beta \ \text{neperunit} \ \Leftrightarrow \ r = e^\beta \ \text{unit} \ . \tag{15}$$

Armed with this baggage, the reader should be able to understand a graphic like the one in figure 1, or to understand this rephrasing of Stephen Hawking's *A Brief History of Time*: At (logarithmic) time T^* = -35 neperseconds, the quaks and antiquarks ... at time T^* = -12 neperseconds, the photons ...

Note: bring Hawking's book home.

2.3 Changing units

Note: mention somewhere how we can change units. Taking the example of a pressure, we have

$$\text{Pa} = 10^5 \ \text{bar} \ . \tag{16}$$

Assume that a logarihtmic pressure has the value $p^* = \alpha$ neperpascal . The, as it is easy to see,

$$p^* = \alpha \ \text{neperpascal} = (\alpha + \log \ 10^5) \ \text{neperbar} \approx (\alpha + 11.51) \ \text{neperbar} \ . \tag{17}$$

This shows that *relative* values of logarithmic parameters (difference between two values) will be independent from the unit.

Note: mention somewhere that, when taking the reciprocal parameter (i.e., for instance, from period to frequency) we have the equivalence:

$$T^* = \alpha \text{ nepersecond} \quad \Leftrightarrow \quad \nu^* = -\alpha \text{ neperhertz} . \tag{18}$$

2.4 Using binary or decimal logarithms

The use of particular systems of numbering, base 2 in computers, base 10 in ordinary life, make sometimes preferable the use of logarithms with basis other than e . We show in this section that we can conciliate the need of defining logarithmic parameters using only natural logarithms and the practical use of base 2 or bare 10 logarithms.

Let us take the example of a frequency, the generalization being obvious. As, for any base a , $\log x = log_a x \cdot \log a$, given the constant ν_0 , the logarithmic frequency can be written, for any logarithm basis,

$$\nu^\star = \log \frac{\nu}{\nu_0} = \left(\log_a \frac{\nu}{\nu_0}\right) \cdot \log a . \tag{19}$$

In particular,

$$\nu^\star = \log \frac{\nu}{\nu_0} = \left(\log_2 \frac{\nu}{\nu_0}\right) \cdot \log 2 = \left(\log_{10} \frac{\nu}{\nu_0}\right) \cdot \log 10 . \tag{20}$$

We need now an important definition. We have seen that, in fact, a neperhertz is, like a radian, just the number 1. Let us define the following numerical constants:

$$\begin{aligned} \text{neperhertz} &= \log e = 1 \\ \text{boolehertz} &= \log 2 \approx 0.693 \\ \text{belhertz} &= \log 10 \approx 2.30 . \end{aligned} \tag{21}$$

Then, equation 20 can be rewritten

$$\nu^\star = \left(\log \frac{\nu}{\nu_0}\right) \text{ neperhertz} = \left(\log_2 \frac{\nu}{\nu_0}\right) \text{ boolehertz} = \left(\log_{10} \frac{\nu}{\nu_0}\right) \text{ belhertz} . \tag{22}$$

Thanks to the numerical constants introduced, a logarithmic parameter, defined using exclusively using natural logarithms, can be practically computed using base 2 or base 10 logarithms.

As an example, we have considered, in section 2.1, a logarithmic frequency of $\nu^* = 9.42 \text{ neperhertz}$. We can write $\nu^* = 9.42 \text{ neperhertz} = 13.6 \text{ boolehertz} = 4.09 \text{ belhertz}$.

Note: say somewhere that the term “boole” comes, of course, from the algebra in base 2, and that the term “bel” is chosen in reference to the bel and decibel of engineers (I will show below the equivalence).

We have the simple properties

$$\begin{aligned} \nu^* = \alpha \text{ neperhertz} &\quad \Leftrightarrow \quad \nu = e^\alpha \text{ hertz} \\ \nu^* = \beta \text{ boolehertz} &\quad \Leftrightarrow \quad \nu = 2^\beta \text{ hertz} \\ \nu^* = \gamma \text{ belhertz} &\quad \Leftrightarrow \quad \nu = 10^\gamma \text{ hertz} . \end{aligned} \tag{23}$$

While the booleunit indicates the power of two expressing the value of the original (non logarithmic) parameter, the belunit indicates the power of ten.

For the sake of completeness, we can rewrite equation 23 as

$$\begin{aligned} \nu = \varepsilon \text{ hertz} \quad &\Leftrightarrow \quad \nu^\star = (\log \varepsilon) \text{ neperhertz} \\ &\Leftrightarrow \quad \nu^\star = (\log_2 \varepsilon) \text{ boolehertz} \\ &\Leftrightarrow \quad \nu^\star = (\log_{10} \varepsilon) \text{ belhertz} \ . \end{aligned} \tag{24}$$

Note that the constants introduced in equation 21 can, equivalently, be defined by

$$\begin{aligned} \text{neperhertz} &= \log\ e = 1 \\ \text{boolehertz} &= (\log 2) \text{ neperhertz} \approx 0.693 \text{ neperhertz} \\ \text{belhertz} &= (\log 10) \text{ neperhertz} \approx 2.30 \text{ neperhertz} \ . \end{aligned} \tag{25}$$

Note that while the parameter

$$\nu^* = \log \frac{\nu}{\nu_0} \ ,$$

defined as a natural logarithm, gets the name of *logarithmic frequency*, the quantities

$$\nu^*_2 = \log_2 \frac{\nu}{\nu_0}$$

and

$$\nu^*_{10} = \log_{10} \frac{\nu}{\nu_0}$$

appearing in equation 22 do not receive a special name, as its direct use is proscribed. Caution: for any $a \neq e$,

$$\nu^\star_a = (\log\ a)\ \nu^\star \neq \nu^\star \ . \tag{26}$$

2.5 The "decibel"

Note: this section has to be rewritten, as I have already introduced the relative measurements above.

Let us consider a quasi-sinusoidal acoustic wave. Let p be the amplitude of the pressure, and let i be its intensity (power per unit of surface). It is well known that they are related by

$$i = \frac{p^2}{2\rho c} \ , \tag{27}$$

where ρ is the volumetric mass of the medium and c the speed of the waves.

The logarithmic amplitude and intensity are

$$p^\star = \log \frac{p}{p_0} \quad ; \quad i^\star = \log \frac{i}{i_0} \ , \tag{28}$$

where for the constants p_0 and i_0 we can take the IS units (Pa and $\mathrm{W\,m^{-2}}$), if the we wish to have "absolute" measurements. We have

$$i^\star = \log \frac{i}{i_0} = \log \left(\frac{p}{p_0}\right)^2 = 2\ \log \frac{p}{p_0} = 2\ p^\star \ , \tag{29}$$

i.e., if we have

$$p^\star = \alpha \text{ neperpascal} \ , \tag{30}$$

then,

$$i^\star = 2\ \alpha \text{ neper} - (\text{watt} - \text{per} - \text{squared} - \text{meter}) \ . \tag{31}$$

This factor of 2, although natural here, is at the center of some confusions when using the decibel scale of eengineers (see below).

Going now to relative measurements, imagine that from the two pressure amplitudes p_A and p_B (resp. from the two intensities i_A and i_B) we have defined

$$i^\star_A = \log\frac{i_A}{i_0} \quad ; \quad p^\star_A = \log\frac{p_A}{p_0} \quad ; \quad i^\star_B = \log\frac{i_B}{i_0} \quad ; \quad p^\star_B = \log\frac{p_B}{p_0} \ . \tag{32}$$

The relative values $\Delta p^\star = p^\star_B - p^\star_A$ and $\Delta i^\star = i^\star_B - i^\star_A$ are easily seen to equal

$$\Delta p^\star = \log\frac{p_B}{p_A} \quad ; \quad \Delta i^\star = \log\frac{i_B}{i_A} \ . \tag{33}$$

We have, as before,

$$\Delta i^\star = 2\ \Delta p^\star \ . \tag{34}$$

It is important that $\Delta p^\star$ and $\Delta i^\star$ are independent on the original values of p_0 and i_0 allowing the absolute mesurements. Therefore, I propose:

1. keeping, for these relative measurements, a notation with the Δ , to avoid confusing with absolute values;

2. as the relative values are independent of the units chosen, use the terminology "the relative logarithmic pressure is $\Delta p^\star = \alpha$ *neperpressure*" or "the relative logarithmic intensity is $\Delta i^\star = \beta$ *neperintensity*".

In addition to the neper, we, of course, also can introduce the boole and the bel as in equation 25. Then, if in a relative measurement for an acoustic wave we have, for instance,

$$\Delta p^\star = 7.0 \text{ neperpressure} \ , \qquad \text{or, equivalently,} \qquad \Delta i^\star = 14.0 \text{ neperintensity} \ , \tag{35}$$

using the bel instead of the neper gives (according to definition 25)

$$\Delta p^\star = 3.04 \text{ belpressure} \ , \qquad \text{or, equivalently,} \qquad \Delta i^\star = 6.08 \text{ belintensity} \ , \tag{36}$$

or, using the decibel instead of the bel,

$$\Delta p^\star = 30.4 \text{ decibelpressure} \ , \qquad \text{or, equivalently,} \qquad \Delta i^\star = 60.8 \text{ decibelintensity} \ . \tag{37}$$

This is exactly equivalent to the dB scale of engineers.

The clarification of this odd scale has required to use a Δ notation, to add "pressure", "intensity" (or whatever other parameter), and, most importantly, to give, in fact, to the "decibel" the numerical value $\frac{1}{10}\log 10$ (see definitions 25).

Note: say somewhere that, sometimes, noise levels are measures with reference to some "normal-conversation-pressure" (ncp) or the equivalent "normal-conversation-intensity" (nci). The ncp is taken as $10^{-1} Pa$. It is then onvious what the "decibel-ncp" or the "decibel-nci" can be.

With temperatures, one sometimes uses the "normal temperature" (nt). One then could introduce, for instance, the nepper-nt.

3 Analysis of uncertainties

Note: say here that the parameters considered in this paper lead, usually, to consider the *relative* measurement uncertainties (they are almost independent of the measured value). Note: explain better. These uncertainties are transformed into ordinary (absolute) uncertainties for the logarithmic parameters. Note: explain better.

Note: say that the Gaussian model for measurement uncertainties can be used only on the logarithmic parameter.

4 The manufacture of measuring instruments

Today, measurement apparatus are designed to measure parameters like electric resistences or frequencies. I will claim now that these should be manufactured differently, in order to deliver results expressing directly the logarithmic magnitudes.

Note: say here that an instrument made to measure the electric resistence of a conducting wire today is made to give the results in dofferent "scales", that may range, for instance, from the milliohm, $m\Omega$, to the megaohm, $M\Omega$. If manufactured to indicate the logarithmic resistence would indicate, if using decimal logarithms, values ranging from -3 belohm to 6 belhom, or, using natural logarithms, from -7 neperohm to 14 neperohm.

Note: my intuition says that such instruments would ne easier to manufacture that present day instruments. Check this.

Figure 2 suggests the behaviour of a "neper-ohm-meter".

Note: I remember having seen, on my youth, analogical meters in the musical equipments (measuring perhaps the output power) where the needle was indicating a value in a logarithmic scale. This suggests that an equivalent digital measuring instrument would be more simple to manufacture in indicating the logarithm parameter.

5 Music

I have stressed enough that a logarithmic frequency is a more "linear" parameter than a frequency. This, is, of course, obvious for musicians, who, when passing from an octave to the next, just double the frequencies of the sounds.

Disregarding here older scales (like the Pythagorean scale or the Zarlino one), the tempered (?) scale of Werckmeister (1645–1706), popularized by Johann Sebastian Bach (1685–1750), and the one widely used today, corresponds to a logarithmic progression of the frequencies of the notes.

L. Euler (Testamen Novae Theoriae Musicae, ibid., 1974) suggested using logarithms in base $\sqrt[12]{2} \approx 1.059463$ to measure musical intervals (so the reader can imagine what I would call an eulerhertz). In this scale, the succesive half-tones can be made to correspond to the successive integers.

Without taking a so extreme view of things, the use of logarithms in base 2, gives a sufficiently simple scheme. Figure 3 gives the frequencies and the logarithmic frequencies for three octaves of the musical notes. For completeness, I give the logarithmic frequencies in three units, the neperhertz, the boolehertz and the belhertz. The boolehertz unit is simpler in what the half-tone interval makes exactly 1/12 of boolehertz. Passing from one octave to the next, increases the logarithmic frequency of one boolhertz.

This suggests the simplification of the musical scale shown in figure 4, where the pitch of the UT 3 has been fixed at the exact value of 8 boolehertz (256 hertz). Then, all the UT notes have pitches corresponding to the successive integers in the boolehertz scale, i.e., to the succesive powers of 2 in the hertz scale. In this modified scale, the LA 3 has its frequency shifted from the nominal 440 hz to 430.54 hz. This would give a numerical basis for a general descent of the pitch that is desired, for pure musical reasons, by many mucisians (note: give a recent reference here) (note: explain that a UNESCO commission is presently studying this problem of definition of the nominal pitch for the scale).

6 More complex parameters

Note: Talk here about the Poisson's ratio and the Hubble constant.

7 Conclusion

The original unit is the nonlinear one. The "logarithmic" unit is natural. And linear.

Note: try to make obvious that I am making more than using logarithmic scales. I try to give life to the logarithmic parameters.

8 Appendix

8.1 Exact values

$$
\begin{aligned}
1 \text{ neperunit} &= (\log_{10} e) \text{ belunit} = \frac{1 \text{ belunit}}{\log 10} = (\log_2 e) \text{ booleunit} = \frac{1 \text{ booleunit}}{\log 2} \\
1 \text{ booleunit} &= (\log 2) \text{ neperunit} = \frac{1 \text{neperunit}}{\log_2 e} = (\log_{10} 2) \text{ belunit} = \frac{1 \text{belunit}}{\log_2 10} \\
1 \text{ belunit} &= (\log_2 10) \text{ booleunit} = \frac{1 \text{ booleunit}}{\log_{10} 2} = (\log 10) \text{ neperunit} = \frac{1 \text{ neperunit}}{\log_{10} e} \ .
\end{aligned}
$$

8.2 Numerical values

$$
\begin{aligned}
1 \text{ neperunit} &\approx 0.434 \text{ belunit} \approx 1.44 \text{ booleunit} \\
1 \text{ booleunit} &\approx 0.693 \text{ neperunit} \approx 0.301 \text{ belunit} \\
1 \text{ belunit} &\approx 3.32 \text{ booleunit} \approx 2.30 \text{ neperunit} \ . \qquad (38)
\end{aligned}
$$

9 References and Notes

1 Jeffreys, H., 1939, Theory of probability, Clarendon Press, Oxford.

2 Jaynes, E.T., 1995, Probability theory: the logic of science, Internet (ftp: bayes.wustl.edu).

3 Tarantola, A., 1987, Inverse problem theory; methods for data fitting and model parameter estimation, Elsevier; Tarantola, A., 1990, Probabilistic foundations of Inverse Theory, in: *Geophysical Tomography*, Desaubies, Y., Tarantola, A., and Zinn-Justin, J., (eds.), North Holland.

4 Tarantola, A., and Valette, B., 1982, Inverse Problems = Quest for Information, *J. Geophys.*, 50, 159-170.

5 Shannon, C.E., 1948, A mathematical theory of communication, Bell System Tech. J., 27, 379–423.

6 Cook, A., 1994, The observational foundations of physics, Cambridge University Press.

7 For instance, they can be taken equal to the standards of time duration and of frequency, 9 192 631 770 Hz and (1/9 192 631 770) s respectively.

8 There is an amusing consequence to the fact that it is the logarithm of the length (or the surface, or the volume) of an object that is the natural (i.e., Cartesian) variable. The Times Atlas of the World (comprehensive edition, Times books, London, 1983) starts by listing the surfaces of the states, territories, and principal islands of the world. The interesting fact is that the *first digit* of the list is far from having an uniform distribution in the range 1–9: the observed frequencies closely match the probability $p(n) = \log_{10}((n+1)/n)$, (i.e., 30% of the occurrences are 1's, 18% are 2's,. . . , and less than 5% are 9's), that is the theoretical distribution one should observe for a parameter whose probability density is of the form $1/x$. A list using the logarithm of the surface should not present this effect, and all the digists 1–9 would have the same probability for appearing as first digit. This effect explains the amusing fact first reported by Frank Benford in 1939: that the books containing tables of logarithms (used, before the advent of digital computers, to make computations) have usually their first pages more damaged by use than their last pages. . .

9 Guide to the expression of uncertainty in measurement, International Organization of Standardization (ISO), Switzerland, 1993. B.N. Taylor and C.E. Kuyatt, 1994, Guidelines for evaluating and expressing the uncertainty of NIST measurement results, NIST technical note 1297.

10 We thank . . . Stefan Nielsen . . . Enrique Zamora. . .

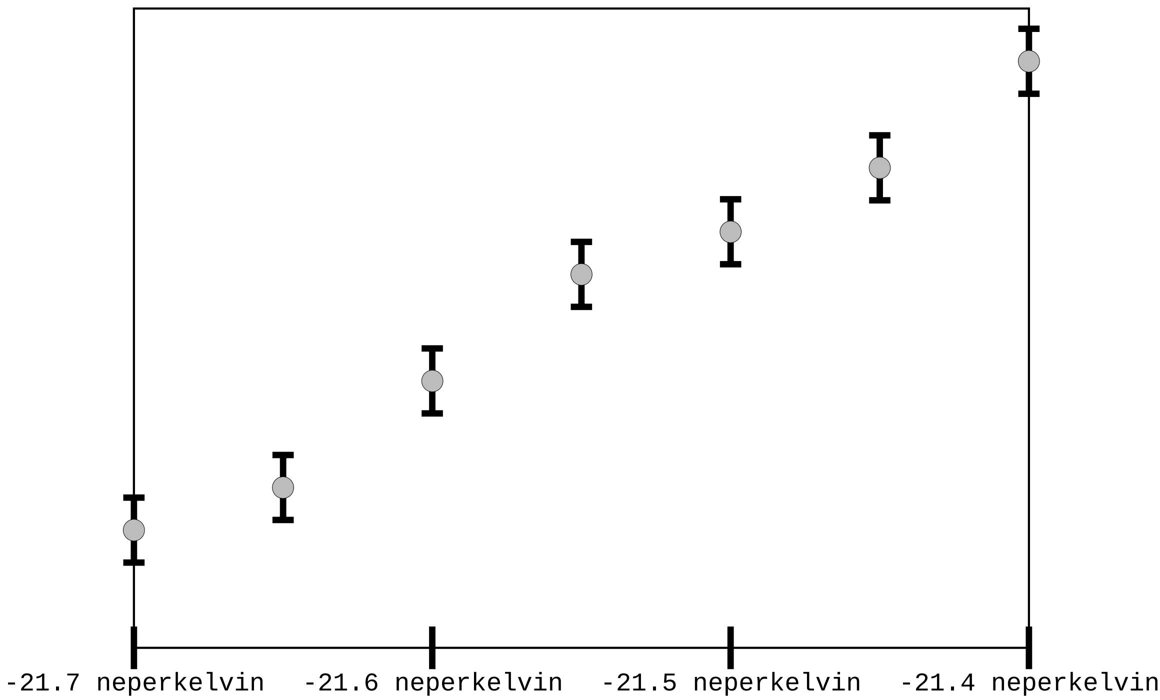

Figure 1: A small exercise of interpreation to check if the reader has assimilated the notion of logarithmic temperature.

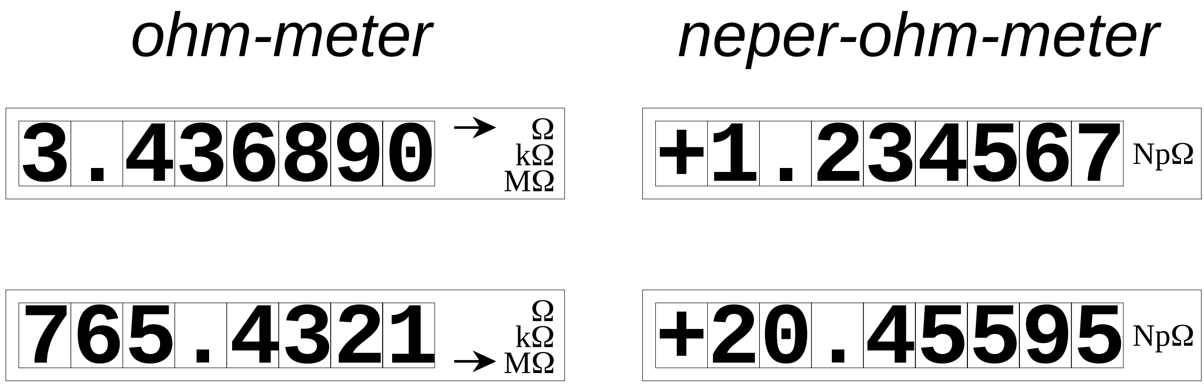

Figure 2: Today, measuring instruments are manufactured so that one obtains ohms, hertzs, pascals... As explained in the text, these parameters are not natural (the probability density representing the state of total ignorance on the value of the parameter [to represent, for instance, the result of an infitely bad measurement] is not constant). The logarithm of these parameters is the natural parameter to use. Then, the algebra of analysis of uncertainties is simplified (absolute errors replace relative errors). I propose to directly manufacture measuring instruments using the logarithmic parameters. One additional advantage of these instruments would be that, while the intrinsic exponential character of parameters like an electric resistence, a frequency, a pressure..., forces to make changes of scale (say, from the milli-ohm scale to the mega-ohm scale), the natural parameters can accomodate any realistic value on the same scale (note that $\log(10^{-40}) = -92.1$ and that $\log(10^{+40}) = +92.1$). Here, at left, a classical instrument makes two measurements, using, at top the ohm scale and, at bottom, the megaohm scale. At right, a logarithmic instrument has a single scale, measuring neperohms (defined as $R^* = \log(R/R_0)$, with $R_0 = 1\Omega$). Probably, engineers will prefer base 10 logarithms. Then, the belohm can be used, where $1\,\text{belohm} \approx 2.30\,\text{neperohm}$. The two readings at right then would be $R_1^* = +0.536166 \quad \text{belohm}$ and $R_2^* = +8.883907 \quad \text{belohm}$. As explained in the text, these definitions reconcile the use of a “bel” or “decibel” scale with the necessary generality and simplicity of physics.

tone	frequency	logarithmic frequency		
C UT	1046.50 hz	6.9532..... neperhertz =	......59714 boolehertz =	.... belhertz
SI	987.77 hz	6.8954..... neperhertz =	......26381 boolehertz =	.... belhertz
B LA#	932.33 hz	6.8377..... neperhertz =	9.864693047 boolehertz =	2.969568505 belhertz
A LA	880.00 hz	6.779921907 neperhertz =	9.781359714 boolehertz =	2.944482672 belhertz
SOL#	830.61 hz	6.7222..... neperhertz =	9.698026381 boolehertz =	2.919396839 belhertz
G SOL	783.99 hz	6.6644..... neperhertz =	9.614693047 boolehertz =	2.894311006 belhertz
FA#	739.99 hz	6.6066..... neperhertz =	9.531359714 boolehertz =	2.869225173 belhertz
F FA	698.46 hz	6.5489..... neperhertz =	9.448026381 boolehertz =	2.844139340 belhertz
E MI	659.26 hz	6.4911..... neperhertz =	9.364693047 boolehertz =	2.819053507 belhertz
RE#	622.25 hz	6.4333..... neperhertz =	9.281359714 boolehertz =	2.793967674 belhertz
D RE	587.33 hz	6.3756..... neperhertz =	9.198026381 boolehertz =	2.768881841 belhertz
UT#	554.37 hz	6.3178..... neperhertz =	9.114693047 boolehertz =	2.743796008 belhertz
C UT	523.25 hz	6.260061521 neperhertz =	9.031359714 boolehertz =	2.718710175 belhertz
SI	493.88 hz	6.202299256 neperhertz =	8.948026381 boolehertz =	2.693624342 belhertz
B LA#	466.16 hz	6.1445..... neperhertz =	8.864693047 boolehertz =	2.668538509 belhertz
A LA	440.00 hz	6.086774727 neperhertz =	8.781359714 boolehertz =	2.643452676 belhertz
SOL#	415.30 hz	6.0290..... neperhertz =	8.698026381 boolehertz =	2.618366844 belhertz
G SOL	392.00 hz	5.9713..... neperhertz =	......93047 boolehertz =	.... belhertz
FA#	369.99 hz	5.9135..... neperhertz =	......59714 boolehertz =	.... belhertz
F FA	349.23 hz	5.8557..... neperhertz =	......26381 boolehertz =	.... belhertz
E MI	329.63 hz	5.7980..... neperhertz =	......93047 boolehertz =	.... belhertz
RE#	311.13 hz	5.7402..... neperhertz =	......59714 boolehertz =	.... belhertz
D RE	293.66 hz	5.6824..... neperhertz =	......26381 boolehertz =	.... belhertz
UT#	277.18 hz	5.6247..... neperhertz =	......93047 boolehertz =	.... belhertz
C UT	261.63 hz	5.5669..... neperhertz =	......59714 boolehertz =	.... belhertz
SI	246.94 hz	5.5092..... neperhertz =	......26381 boolehertz =	.... belhertz
B LA#	233.08 hz	5.4514..... neperhertz =	......93047 boolehertz =	.... belhertz
A LA	220.00 hz	5.393627546 neperhertz =	......59714 boolehertz =	.... belhertz
SOL#	207.65 hz	5.3359..... neperhertz =	......26381 boolehertz =	.... belhertz
G SOL	196.00 hz	5.2781..... neperhertz =	......93047 boolehertz =	.... belhertz
FA#	185.00 hz	5.2203..... neperhertz =	......59714 boolehertz =	.... belhertz
F FA	174.61 hz	5.1626..... neperhertz =	......26381 boolehertz =	.... belhertz
E MI	164.81 hz	5.1048..... neperhertz =	......93047 boolehertz =	.... belhertz
RE#	155.56 hz	5.0471..... neperhertz =	......59714 boolehertz =	.... belhertz
D RE	146.83 hz	4.9893..... neperhertz =	......26381 boolehertz =	.... belhertz
UT#	138.59 hz	4.9315..... neperhertz =	......93047 boolehertz =	.... belhertz
C UT	130.81 hz	4.8738..... neperhertz =	......59714 boolehertz =	.... belhertz

half-tone interval = log_e(2)/12 neperhertz = 0.05776226505 neperhertz
= log_2(2)/12 boolehertz = 1/12 boolehertz = 0.08333333333 boolehertz
= log_10(2)/12 belhertz = 0.02508583297 belhertz

Figure 3: The tempered (?) scale of musical tones, expressed in frequencies and in logarithmic frequencies, using the three more usual logarithmic scales, natural, binary, and decimal logarithms.

tone	frequency	logarithmic frequency
C UT	1024 hz	10 boolehertz
SI	966.53 hz	$9.91\overgroup{66}$ boolehertz
B LA#	912.28 hz	$9.83\overgroup{33}$ boolehertz
A LA	861.08 hz	9.75 boolehertz
SOL#	812.75 hz	$9.66\overgroup{66}$ boolehertz
G SOL	767.13 hz	$9.58\overgroup{33}$ boolehertz
FA#	724.08 hz	9.50 boolehertz
F FA	683.44 hz	$9.41\overgroup{66}$ boolehertz
E MI	645.08 hz	$9.33\overgroup{33}$ boolehertz
RE#	608.87 hz	9.25 boolehertz
D RE	574.70 hz	$9.16\overgroup{66}$ boolehertz
UT#	542.45 hz	$9.08\overgroup{33}$ boolehertz
C UT	512 hz	9 boolehertz
SI	483.26 hz	$8.91\overgroup{66}$ boolehertz
B LA#	456.14 hz	$8.83\overgroup{33}$ boolehertz
A LA	430.54 hz	8.75 boolehertz
SOL#	406.37 hz	$8.66\overgroup{66}$ boolehertz
G SOL	383.57 hz	$8.58\overgroup{33}$ boolehertz
FA#	362.04 hz	8.5 boolehertz
F FA	341.72 hz	$8.41\overgroup{66}$ boolehertz
E MI	322.54 hz	$8.33\overgroup{33}$ boolehertz
RE#	304.44 hz	8.25 boolehertz
D RE	287.35 hz	$8.16\overgroup{66}$ boolehertz
UT#	272.22 hz	$8.08\overgroup{33}$ boolehertz
C UT	256 hz	8 boolehertz
SI	241.63 hz	$7.91\overgroup{66}$ boolehertz
B LA#	228.07 hz	$7.83\overgroup{33}$ boolehertz
A LA	215.27 hz	7.75 boolehertz
SOL#	203.19 hz	$7.66\overgroup{66}$ boolehertz
G SOL	191.78 hz	$7.58\overgroup{33}$ boolehertz
FA#	181.02 hz	7.5 boolehertz
F FA	170.86 hz	$7.41\overgroup{66}$ boolehertz
E MI	161.27 hz	$7.33\overgroup{33}$ boolehertz
RE#	152.22 hz	7.25 boolehertz
D RE	143.68 hz	$7.16\overgroup{66}$ boolehertz
UT#	135.61 hz	$7.08\overgroup{33}$ boolehertz
C UT	128 hz	7 boolehertz

half-tone interval = 1/12 boolehertz

Figure 4: A suggestion to modify the musical pitches, in order to simplify the pitch mesurements. The pitch of the UT 3 has been fixed at the exact value of 8 boolehertz (256 hertz). Then, all the UT notes have pitches corresponding to the successive integers in the boolehertz scale, i.e., to the succesive powers of 2 in the hertz scale. In this modified scale, the LA 3 has its frequency shifted from the nominal 440 hz to 430.54 hz. This would give a numerical basis for a general descent of the pitch that is desired, for pure musical reasons, by many mucisians (note: give a recent reference here) (note: explain that a UNESCO commission is presently studying this problem of definition of the nominal pitch for the scale).

Contents