Mantle lateral variations and elastogravitational deformations – I. Numerical modelling

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SUMMARY

The Earth response (deformation and gravity) to tides or to surface loads is traditionally computed assuming radial symmetry in stratified earth models, at the hydrostatic equilibrium. The present study aims at providing a new earth elastogravitational deformation model which accounts for the whole complexity of a more realistic earth.

The model is based on a dynamically consistent equilibrium state which includes lateral variations in density and elastic parameters, and interface topographies. The deviation from the hydrostatic equilibrium has been taken into account as a first-order perturbation. We use a finite element method (spectral element method) and solve numerically the gravitoelasticity equations.

As a validation application, we investigate the deformation of the Earth to surface loads. We first evaluate the classical loading Love numbers with a relative precision of about 0.3 per cent for PREM earth model. Then we assume an ellipsoidal homogeneous incompressible earth with hydrostatic pre-stresses. We investigate the impact of ellipticity on loading Love numbers analytically and numerically. We validate and discuss our numerical model.

Key words: elastogravitational theory, finite element methods, lateral heterogeneity, loadings, tides.

1 INTRODUCTION

At periods greater than 1 hr, the solid earth is mainly deformed by luni-solar tides and by surface loads induced by different external fluid layers (ocean, atmosphere, continental hydrology, ice volumes). This work is devoted to the analytical and numerical development to compute the response of the Earth to such forcing.

The body tides have been investigated since the 19th century. In 1862, Lord Kelvin (Sir William Thomson) made the first calculation of the elastic deformation of a homogeneous incompressible earth under the action of the tidal gravitational potential (Thomson 1862). Some years later, Love (1911) studied a compressible homogeneous earth model and showed that the tidal effects could be represented by a set of dimensionless numbers, the so-called Love numbers. Takeuchi (1950) obtained a first estimation of the Love numbers by a numerical integration of the equations using a reference earth model deduced from seismology. These results have been later extended (Smith 1974; Wahr 1981) to an ellipsoidal, rotating Earth with hydrostatic pre-stresses and a liquid core, and finally the effects of mantle anelasticity have been included (Wahr & Bergen 1986; Dehant 1987).

In addition to tidal forces, mass changes in the atmosphere cause deformation and mass redistribution inside the planet. The Earth's response to such forcing involves both local and global surface motions and variations in the gravity field, which may be observed in geodetic experiments. These hydrological, atmospheric or oceanic effects on the Earth's gravity field are usually modelled for a spherical Earth with hydrostatic pre-stress (e.g. Farrell 1972; Wahr *et al.* 1998), generally identified to the preliminary reference earth model (PREM) developed by Dziewonski & Anderson (1981).

However, the internal structure of the Earth is more complex than in a spherical non-rotating elastic isotropic (SNREI) earth model like PREM. Seismology and fluid dynamic studies show that the mantle presents heterogeneous structure induced by a thermochemical convection (Davaille 1999; Gu *et al.* 2001; Forte & Mitrovica 2001) and a bias from hydrostatic state. Large lateral heterogeneities have taken place on a million year timescale (Courtillot *et al.* 2003), like the two supposed superplumes under the Pacific and South Africa superswells, or like descending slabs. These aspects of the mantle structure are classically not taken into account in the deformation models.

The elastogravitational deformations are presently observed with very high accuracy. The accuracy of superconducting gravimeter and of positioning techniques (GPS, VLBI) has seen a large improvement in the last decade. Moreover, the global gravity field will be of interest in the next 10 yr with the launch of the missions GRACE (in 2002) and GOCE (in 2007), which are dedicated to gravimetry and gradiometry

measurements. Are the present reference earth deformation models sufficiently realistic to correct and to understand the coming deformation and gravity data? One purpose of this work is to answer to this question.

A few studies investigated some aspects of the influence of lateral heterogeneities on the Earth's tidal deformations. Molodenskiy (1977) was the first to address this problem. He investigated a variational approach of the elastogravitational equations and their first-order perturbations induced by lateral variations. Following this approach, Wang (1991) computed a model of the earth solid tides with theoretical lateral variations of density and of rheological parameters of degree 2 or 3. Then Dehant *et al.* (1999) studied the influence of the non-hydrostatical ellipticity of internal boundaries on solid tides (see also Defraigne *et al.* 1996; Defraigne 1997). These different studies showed that the effect of low degree lateral variations on solid tides is small but not necessarily negligible with regard to present gravimeter data accuracy. A few other interesting studies of viscoelastic deformation modelling for a 3-D earth was conducted recently by Zhong *et al.* (2003) and Latytchev *et al.* (2005a,b). Yet they did not take into account possible deviatoric pre-stresses whose effects on the Earth's deformations are totally unknown.

Taking into account the lateral variations in the calculation is a difficult problem as heterogeneities induce a bias from hydrostatic equilibrium in the Earth. The generalized formulation of elastogravitational equations is therefore much more complicated. The elastic stress tensor that has to be used is no more the classical symmetrical Cauchy stress tensor, but the Piola–Kirchhoff tensor which is a heavy object to manipulate (see Valette 1986; Dahlen & Tromp 1998). In order to avoid this problem, one can treat the problem using the first-order perturbation theory. The lateral variations of density and rheological parameters, the interface topographies, and the deviatoric pre-stresses are introduced as small perturbations of a SNREI earth model with a hydrostatical state of pre-stress.

The elastogravitational system of equations is classically solved using spherical harmonics expansion. This approach is convenient for an ellipsoidal earth model with hydrostatic pre-stress state. Taking into account lateral variations within the mantle and the crust would lead to many couplings between the spherical harmonics, which would make the calculation very heavy (even more for short wavelengths lateral variations). We use here a numerical approach, the spectral element method (Komatitsch & Tromp 2002a; Chaljub *et al.* 2003). The efficiency of this method is mostly independent of the presence of lateral variations, and for this reason it is much more adapted to our problem.

This paper presents the numerical model and its validation. A companion paper presents a geophysical application: the impact of different Mantle heterogeneities on M2 body tides (Métivier *et al.* 2006). The present paper is divided in three parts. The first part is dedicated to elastogravitational theory and to the first-order perturbation of the equations. In the second part, we present the variational formulation of the equations and the spectral element method used to solve the elastogravitational equations. Finally, in the last part of the paper, we present a validation. We investigate the effect of hydrostatic ellipticity on earth loading deformation. In this part, the Earth is considered to be homogeneous and incompressible in order to compare our numerical solution with an analytical one.

2 THE THEORY OF ELASTOGRAVITY AND THE LATERAL VARIATIONS

We use the perturbation theory to calculate the elastogravitational deformation of a heterogeneous aspherical Earth. For this approach, we define four states of the earth models (see Fig. 1), following the formalism and the notations of Woodhouse & Dahlen (1978) and Dahlen & Tromp (1998).

2.1 The elastogravitational deformation of a SNREI earth model

2.1.1 The reference earth

Let us define a SNREI earth model composed of fluid and solid layers, in which the physical properties continuously and smoothly vary with radial position. The different layers are delimited by spherical surfaces, called internal boundaries, and the global model is bounded by an external surface.

Throughout the paper this model will be called the reference earth state or initial earth. We will denote by Ω the bounded open set that characterizes the global volume of the planet, and by $\partial\Omega$ its external frontier. We denote by Σ the set of all the internal boundaries. We have $\Sigma = \Sigma_{SS} \cup \Sigma_{SF}$, with Σ_{SS} the set of solid–solid boundaries and Σ_{SF} the set of solid–fluid boundaries.

Let us define a Cartesian coordinate system with origin at the Earth's centre of mass. In a SNREI earth model, the centre of mass and the centre of the sphere (centre of figure) are merged. We denote by $\vec{x}$ the position vector in this system. Each particle of the planet Ω in its unperturbed reference state is characterized by a density ρ_o, a stress tensor $\overline{\overline{\mathbf{T}}}_o$ and a local gravity $\vec{g}_o = -\vec{\nabla}\phi_o$ (ϕ_o is the gravitational potential).

The mechanical and gravitational equilibrium equations governing the state of the reference earth are:

$$\vec{\nabla} \cdot \overline{\overline{\mathbf{T}}}_o = \rho_o \vec{\nabla}\phi_o \quad \text{Mechanical hydrostatic equilibrium,} \tag{1}$$

$$\Delta\phi_o = 4\pi G \rho_o \quad \text{Poisson equation.} \tag{2}$$

In a SNREI model the equilibrium state is assumed to be hydrostatic, that is, with a pre-stress tensor which only depends on the local pressure $\overline{\overline{\mathbf{T}}}_o = -p_o \overline{\overline{\mathbf{I}}}$ ($\overline{\overline{\mathbf{I}}}$ is the unit tensor). The planet has a spherical symmetric configuration, and all the parameters exclusively depend on the radial position.
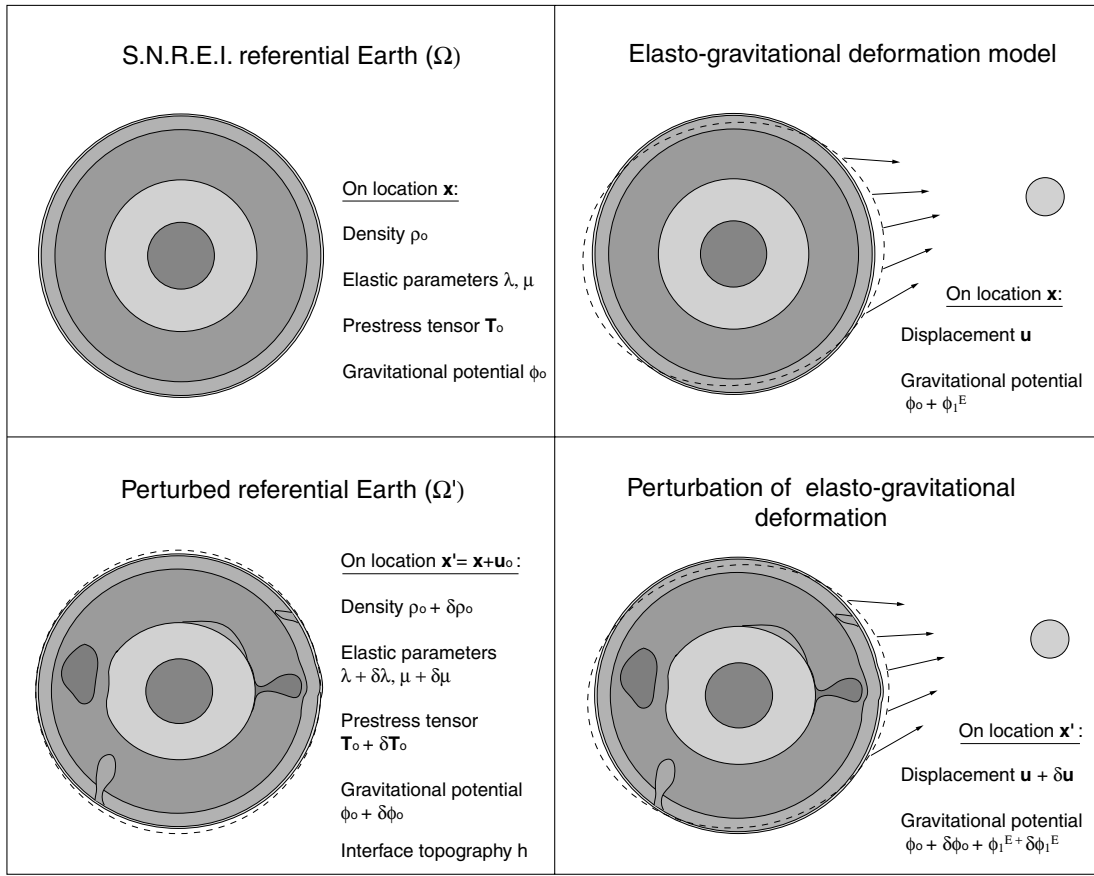
Figure 1. The four distinct states of the planet model defined for the calculation.

2.1.2 Elastogravitational deformations

We now define the mechanical and dynamical deformation of the Earth in response to an external forcing. The planet is subjected to time-dependent gravitational forces per unit volume $\rho_o \vec{f}_{\text{ext}} = -\rho_o \vec{\nabla} V_{\text{ext}}$, where V_{ext} is the external gravitational potential. The gravitational forces can be induced either by luni-solar attraction or by surface loadings. The loadings induce an additionally surface pressure denoted by p_{ext}.

The fundamental assumption in the elastogravitational theory is that the deformations are small compared to the reference configuration. We can thus use a perturbation theory. Each particle of the volume moves along the vector displacement $\vec{u}$, and at the time t, a particle initially located at $\vec{x}$ is now moved to:

$$\vec{r}(\vec{x}, t) = \vec{x} + \vec{u}(\vec{x}, t). \tag{3}$$

We introduce, in a point of the volume, the Eulerian first-order perturbations of density ρ_1^E and of gravitational potential ϕ_1^E, and the Eulerian and Lagrangian first-order perturbations of the Cauchy stress tensor $\overline{\overline{\mathbf{T}}}_1^E$ and $\overline{\overline{\mathbf{T}}}_1^L$, where $\overline{\overline{\mathbf{T}}}_1^E = \overline{\overline{\mathbf{T}}}_1^L + \vec{u} \cdot \vec{\nabla} \overline{\overline{\mathbf{T}}}_o$.

Using the mass conservation equation, one can show that:

$$\rho_1^E(\vec{u}) = -\vec{\nabla} \cdot (\rho_o \vec{u}). \tag{4}$$

The perturbation of the stress tensor in the compressible planet depends on strain and on the relative local displacement of particles. For a SNREI model the constitutive relationship between the Lagrangian Cauchy stress tensor and strain is linear and only depends on the Lamé parameters λ and μ, that is:

$$\overline{\overline{\mathbf{T}}}_1^L(\vec{u}) = \lambda \vec{\nabla} \cdot \vec{u} \, \overline{\overline{\mathbf{I}}} + \mu (\vec{\nabla}\vec{u} + \vec{\nabla}\vec{u}^T). \tag{5}$$

The operation $\vec{\nabla}\vec{u}^T$ is the transposed gradient of $\vec{u}$, so that in component notations $(\vec{\nabla}\vec{u}^T)_{ij} = (\vec{\nabla}\vec{u})_{ji} = \partial_j u_i$, and $\overline{\overline{\mathbf{I}}}$ is the unit tensor. The laws of mechanics at each point $\vec{r}$ of the planet, that is, to momentum equation and Poisson equation at time t are:

$$\frac{d}{dt}\left((\rho_o + \rho_1^E)\,\vec{v}^E\right) - \vec{\nabla} \cdot (\overline{\overline{\mathbf{T}}}_o + \overline{\overline{\mathbf{T}}}_1^E) + (\rho_o + \rho_1^E)\,\vec{\nabla}(\phi_o + \phi_1^E) = -(\rho_o + \rho_1^E)\,\vec{\nabla} V_{\text{ext}}, \tag{6}$$

$$\Delta(\phi_o + \phi_1^E) = 4\pi G \rho_1^E, \tag{7}$$

with $\vec{v}^E = \partial_t \vec{u}$ the Eulerian velocity of particles.

The momentum and Poisson equations are classically expressed in the frame of a linear theory in terms of $\vec{u}$. Subtracting the relation (1) and (2) from equations (6) and (7), we obtain, for all point $\vec{x} \in \Omega$, the classical elastogravitational system composed of the following equations:

$$\rho_o \frac{\partial^2 \vec{u}}{\partial t^2} - \vec{\nabla} \cdot \overline{\overline{\mathbf{T}}}_1^L(\vec{u}) + \vec{\nabla}(\rho_o \vec{u} \cdot \vec{\nabla}\phi_o) - \vec{\nabla} \cdot (\rho_o \vec{u}) \vec{\nabla}\phi_o + \rho_o \vec{\nabla}\phi_1^E = -\rho_o \vec{\nabla} V_{\text{ext}} \tag{8}$$

$$\Delta \phi_1^E + 4\pi G \vec{\nabla} \cdot (\rho_o \vec{u}) = 0. \tag{9}$$

The two independent unknowns are the displacement $\vec{u}$ and the gravitational potential ϕ_1^E (in addition to the related stress-tensor elements). The system is solved with the following boundary conditions, prescribed at any time t:

(i) the surface Eulerian traction must be equal to the external pressure,
(ii) the Eulerian traction has to be continuous through every internal deformed boundary,
(iii) the displacement has to be continuous through every deformed solid–solid or wedged boundary,
(iv) the normal displacement has to be continuous through every deformed solid–fluid boundary,
(v) the Eulerian potential and the normal Eulerian gravity has to be continuous through every deformed surface boundary.

These boundary conditions are applied on boundary surfaces that move and deform with time. The conditions are therefore classically used on the fixed initial spherical boundaries $\partial\Omega \cup \Sigma$ using Lagrangian formalism. And one can show that these continuity relations for a SNREI earth model are simply reduced to (Love 1911; Dahlen & Tromp 1998):

$\partial\Omega$ boundary: $\quad \vec{n}_o \cdot \overline{\overline{\mathbf{T}}}_1^L(\vec{u}) = -p_{\text{ext}} \vec{n}_o,$ \hfill (10)

Σ_{SS} boundaries: $\quad [\vec{u}]_-^+ = 0,$ \hfill (11)

$\qquad\qquad\qquad \left[\vec{n}_o \cdot \overline{\overline{\mathbf{T}}}_1^L(\vec{u})\right]_-^+ = 0,$ \hfill (12)

Σ_{SF} boundaries: $\quad [\vec{n}_o \cdot \vec{u}]_-^+ = 0,$ \hfill (13)

$\qquad\qquad\qquad \left[\vec{n}_o \cdot \overline{\overline{\mathbf{T}}}_1^L(\vec{u})\right]_-^+ = 0,$ \hfill (14)

$\partial\Omega \cup \Sigma$ boundaries: $\quad \left[\phi_1^E\right]_-^+ = 0,$ \hfill (15)

$\qquad\qquad\qquad \left[\vec{n}_o \cdot \vec{\xi}(\vec{u}, \phi_1^E)\right]_-^+ = 0,$ \hfill (16)

with $\vec{\xi}(\vec{u}, \phi_1^E) = \vec{\nabla}\phi_1^E + 4\pi G \rho_o \vec{u}$. We denote by $\vec{n}_o$ the outgoing normal unit vector of each boundary of $\partial\Omega \cup \Sigma$ (unit radial vectors for a SNREI model), and we denote by $[\phi]_-^+ = \phi^+ - \phi^-$ the gap of the variable ϕ through the surface (from the positive side to the negative side of the interface with respect to its outward unit normal vector).

Note on the degree 1 forcing:

For degree 1 deformation, the problem is degenerated. One new condition has to be taken into account in order to preserve the centre of mass position (Mac Cullagh theorem, see Munk & MacDonald 1960). One can show (Farrell 1972; Greff-Lefftz & Legros 1997) that the surface condition for degree 1 gravity potential has to be expressed as: $(\phi_1^E + V_{\text{ext}})_{l=1} = 0$.

2.2 Perturbation of the elastogravitational theory for a realistic earth

2.2.1 The perturbation of the reference earth

We now define the elastogravitational equations for a general heterogeneous aspherical compressible Earth. The approach presented here is based on the perturbation theory. We follow mainly the work developed by Woodhouse & Dahlen (1978) and by Dahlen & Tromp (1998). The perturbation theory has been applied in elastogravitational deformation computation by various authors, we can refer to the works of Smith (1974), Valette (1986), Lognonné & Romanowicz (1990).

Let us define a new reference earth model that contains internal lateral variations of density and of elastic parameters. In this paper, we call this initial model the perturbed reference earth (the reference earth is the SNREI earth previously defined). We denote by Ω' the open set that characterizes the earth volume in the new configuration, and $\partial\Omega'$, $\Sigma' = \Sigma'_{SS} \cup \Sigma'_{SF}$ the equivalent surface boundaries set defined in the previous section, here for this perturbed reference earth model. The surfaces $\partial\Omega' \cup \Sigma'$ are no longer spherical. We denote the topography by h, defined on every point of $\partial\Omega \cup \Sigma$, which corresponds to the normal (radial) displacement of a boundary between the SNREI earth model and the present new reference earth model. One can show then that the normal unit vector of the boundary has been modified by an amount of $\delta\vec{n}_o = -\vec{\nabla}_\Sigma h$ (Dahlen & Tromp 1998) (where we denote $\vec{\nabla}_\Sigma$ as the surface gradient defined on the interface Σ. Note that $\vec{\nabla} = \vec{n}_o \partial_n + \vec{\nabla}_\Sigma$ with the normal gradient $\partial_n = \vec{n}_o \cdot \vec{\nabla}$).

We denote by $\delta\rho_o$, $\delta\lambda$ and $\delta\mu$ the incremental perturbation of density and of Lamé elastic parameters of the initial earth, with respect to the SNREI earth model defined in the last section. The lateral variations within the planet can be induced by the mantle convection for example, or by crustal complex history formation, etc. In general, their existence involve internal complex dynamical phenomena. Therefore, the equilibrium state of the planet cannot be simply hydrostatic. We note the first-order perturbation of pre-stress tensor:

$$\delta\overline{\overline{\mathbf{T}}}_o = -\delta p_o \overline{\overline{\mathbf{I}}} + \delta\overline{\overline{\Sigma}}_o, \tag{17}$$

with δp_o the perturbed local pressure, $\overline{\overline{\mathbf{I}}}$ the identity tensor and $\delta\overline{\overline{\Sigma}}_o$ the deviatoric pre-stress tensor (so that $trace\{\delta\overline{\overline{\Sigma}}_o\} = 0$). At last, we denote by $\delta\phi_o$ the incremental perturbation of gravity potential and $\delta\vec{g}_o$ the incremental perturbation of gravity with respect to the SNREI model.

The new reference planet follows the mechanical and gravitational equilibrium described in the next section. However, using eqs (1) and (2), we have new relations restricted to first order:

$$\vec{\nabla}\cdot\delta\overline{\overline{\mathbf{T}}}_o = \delta\rho_o\vec{\nabla}\phi_o + \rho_o\vec{\nabla}\delta\phi_o, \tag{18}$$

$$\Delta\delta\phi_o = 4\pi G\delta\rho_o. \tag{19}$$

2.2.2 Perturbation of elastogravitational deformations

Under the impact of $\vec{f}_{\rm ext}$ the planet is deformed and obeys the elastogravitational theory in our context. Following mainly the Dahlen & Tromp (1998) notations, we denote by $\delta\vec{u}$ and by $\delta\phi_1^E$ the first-order perturbation of $\vec{u}$ and of ϕ_1^E induced by the modification of the reference earth with respect to the unperturbed SNREI reference earth model (note that $\delta\phi_1^E$, as other new perturbed variables, is a first-order perturbation with respect to ϕ_1^E, but a second-order perturbation with respect to the initial ϕ_o). We define then $\delta\overline{\overline{\mathbf{T}}}_1^L$ the first-order perturbation of the Lagrangian Cauchy stress tensor $\overline{\overline{\mathbf{T}}}_1^L$. This tensor, using the constitutive relation, can be divided into two distinct parts:

$$\delta\overline{\overline{\mathbf{T}}}_1^L = \overline{\overline{\mathbf{T}}}_1^L(\delta\vec{u}) + \delta\overline{\overline{\tau}}(\vec{u}), \tag{20}$$

where $\delta\overline{\overline{\tau}}(\vec{u}) = \delta\lambda\,\vec{\nabla}\cdot\vec{u}\,\overline{\overline{\mathbf{I}}} + \delta\mu(\vec{\nabla}\vec{u} + \vec{\nabla}\vec{u}^T)$.

At the time t, the planet obeys the laws of mechanics and of potential theory. From the classical elastogravitational system of equations (eqs 8 and 9), we can then express a perturbed elastogravitational system which remains in the frame of a linear theory in terms of $\delta\vec{u}$ and $\delta\phi_1^E$. The perturbed system is, for all point $\vec{x} \in \Omega$:

$$\rho_o\frac{\partial^2\delta\vec{u}}{\partial t^2} - \vec{\nabla}\cdot\overline{\overline{\mathbf{T}}}_1^L(\delta\vec{u}) + \vec{\nabla}(\rho_o\delta\vec{u}\cdot\vec{\nabla}\phi_o) - \vec{\nabla}\cdot(\rho_o\delta\vec{u})\vec{\nabla}\phi_o + \rho_o\vec{\nabla}\delta\phi_1^E = \vec{f}_1 \tag{21}$$

$$\Delta\delta\phi_1^E + 4\pi G\vec{\nabla}\cdot(\rho_o\delta\vec{u}) = f_2, \tag{22}$$

with two right-hand side terms defined as:

$$\begin{aligned}\vec{f}_1 = {}& \delta\rho_o(\vec{f}_{\rm ext} - \partial_{tt}\vec{u} - \vec{\nabla}\phi_1^E) + \vec{\nabla}\cdot\delta\overline{\overline{\tau}}(\vec{u}) + \vec{\nabla}(\vec{u}\cdot(\vec{\nabla}\cdot\delta\overline{\overline{\Sigma}}_o)) - \vec{\nabla}\cdot(\vec{u}\cdot\vec{\nabla}\delta\overline{\overline{\Sigma}}_o)\\ & - \vec{\nabla}(\delta\rho_o\,\vec{u}\cdot\vec{\nabla}\phi_o) + \vec{\nabla}\cdot(\delta\rho_o\,\vec{u})\vec{\nabla}\phi_o - \vec{\nabla}(\rho_o\,\vec{u}\cdot\vec{\nabla}\delta\phi_o) + \vec{\nabla}\cdot(\rho_o\,\vec{u})\vec{\nabla}\delta\phi_o\end{aligned} \tag{23}$$

$$f_2 = -4\pi G\vec{\nabla}\cdot(\delta\rho_o\vec{u}). \tag{24}$$

We see here that the new system of eqs (21) and (22) is similar to the classical elastogravitational system of eqs (8)–(9) apart from the right-hand side terms $\vec{f}_1$ and f_2. These functions only depend on the different parameters of the reference earth and on the solutions of the classical elastogravitational system $\vec{u}$ and ϕ_1^E. Consequently, if we know the deformation solutions for a SNREI earth model, the two unknowns are reduced here to $\delta\vec{u}$ and $\delta\phi_1^E$. The great interest of the perturbation theory is that we finally have to solve the system of equations at each point of the SNREI earth model Ω which is spherical. All the information due to asphericity and lateral variations of the parameters (etc.) is included in the right-hand side of the equations.

The general boundary conditions for this calculation are the same as those in the last section conditions but applied here to the new reference earth model interfaces $\partial\Omega' \cup \Sigma'$. With respect to first-order perturbation theory, the boundary conditions are applied to surface boundaries of the SNREI earth model $\partial\Omega \cup \Sigma$. The advantage of the method lies in the fact that the positions of these boundaries are fixed and their shapes are spherical. Nevertheless, the boundary relations appear to be no longer continuous through these surfaces. Let us express them as follows:

$\partial\Omega$ boundary:
$$\vec{n}_o\cdot\overline{\overline{\mathbf{T}}}_1^L(\delta\vec{u}) = \vec{B}_{\partial\Omega}, \tag{25}$$

Σ_{SS} boundaries:
$$[\delta\vec{u}]_-^+ = \vec{B}_u^S, \tag{26}$$

$$\left[\vec{n}_o\cdot\overline{\overline{\mathbf{T}}}_1^L(\delta\vec{u})\right]_-^+ = \vec{B}_{\mathbf{T}}^S, \tag{27}$$

Σ_{SF} boundaries:
$$[\vec{n}_o\cdot\delta\vec{u}]_-^+ = \vec{B}_u^F, \tag{28}$$

$$\left[\vec{n}_o \cdot \overline{\overline{\mathbf{T}}}_1^L(\delta\vec{u})\right]_-^+ = \vec{B}_{\mathbf{T}}^F, \tag{29}$$

Σ_ρ boundaries: $\qquad \left[\delta\phi_1^E\right]_-^+ = B_\phi, \tag{30}$

$$\left[\vec{n}_o \cdot \vec{\xi}(\delta\vec{u}, \delta\phi_1^E)\right]_-^+ = B_\xi. \tag{31}$$

The B_q terms can be calculated separately (q as any symbol). The principle of the calculations is presented in the Appendix A. We present here the final formulations in the different terms:

$$\vec{B}_{\partial\Omega} = -\delta p_{\text{ext}}\,\vec{n}_o + p_{\text{ext}}\vec{\nabla}_\Sigma h - \vec{n}_o \cdot \delta\tau(\vec{u}) - h\,\partial_n\overline{\overline{\mathbf{T}}}_1^L(\vec{u}) \cdot \vec{n}_o + \vec{\nabla}_\Sigma h \cdot \overline{\overline{\mathbf{T}}}_1^L(\vec{u}) + (\delta\overline{\overline{\mathbf{T}}}_o + h\,\partial_n\overline{\overline{\mathbf{T}}}_o) \cdot \vec{\nabla}_\Sigma\vec{u} \cdot \vec{n}_o \tag{32}$$

$$\vec{B}_u^S = -h\,[\partial_n\vec{u}]_-^+. \tag{33}$$

$$\vec{B}_{\mathbf{T}}^S = -\left[\vec{n}_o \cdot \delta\tau(\vec{u}) - h\,\partial_n\overline{\overline{\mathbf{T}}}_1^L(\vec{u}) \cdot \vec{n}_o + \vec{\nabla}_\Sigma h \cdot \overline{\overline{\mathbf{T}}}_1^L(\vec{u}) + (\delta\overline{\overline{\mathbf{T}}}_o + h\,\partial_n\overline{\overline{\mathbf{T}}}_o) \cdot \vec{\nabla}_\Sigma\vec{u} \cdot \vec{n}_o\right]_-^+ \tag{34}$$

$$\vec{B}_u^F = -h\,[\partial_n\vec{u} \cdot \vec{n}_o]_-^+ + \vec{\nabla}_\Sigma h \cdot [\vec{u}]_-^+, \tag{35}$$

$$\vec{B}_{\mathbf{T}}^F = -\left[\vec{n}_o \cdot \delta\tau(\vec{u}) - h\,\partial_n\overline{\overline{\mathbf{T}}}_1^L(\vec{u}) \cdot \vec{n}_o + \vec{\nabla}_\Sigma h \cdot \overline{\overline{\mathbf{T}}}_1^L(\vec{u}) + (\delta\overline{\overline{\mathbf{T}}}_o + \delta\varpi\overline{\overline{\mathbf{I}}}) \cdot \vec{\nabla}_\Sigma\vec{u} \cdot \vec{n}_o - \vec{n}_o\vec{u} \cdot \vec{\nabla}_\Sigma(\delta\varpi + h\,\partial_n\varpi)\right]_-^+ \tag{36}$$

$$B_\phi = -h\,\left[\partial_n\phi_1^E\right]_-^+, \tag{37}$$

$$B_\xi = -h\,\left[\partial_n\vec{\xi}(\vec{u}, \phi_1^E) \cdot \vec{n}_o\right]_-^+ + \vec{\nabla}_\Sigma h \cdot \left[\vec{\xi}(\vec{u}, \phi_1^E)\right]_-^+ - 4\pi G\,[\delta\rho_o]_-^+\,\vec{u} \cdot \vec{n}_o, \tag{38}$$

with $\quad \varpi = -\vec{n}_o \cdot \overline{\overline{\mathbf{T}}}_o \cdot \vec{n}_o, \tag{39}$

and $\quad \delta\varpi = -\vec{n}_o \cdot \delta\overline{\overline{\mathbf{T}}}_o \cdot \vec{n}_o. \tag{40}$

Note on the degree 1 deformation:

For a more complex earth than SNREI model, a degree 1 spheroidal deformation is possible as far as the centre of mass of the whole system remains fixed. For all type of forcing, we need to have the total external potential of degree 1 null (known as the free space potential):

$$\left((\phi_o + \delta\phi_o + \phi_1^E + \delta\phi_1^E)_{\text{ext}} + V_{\text{ext}}\right)_{l=1} = 0. \tag{41}$$

On the reference surface $\partial\Omega$, this relation leads to a new boundary condition for degree 1 of $\delta\phi_1^E$ (using the different boundary conditions for ϕ_o, $\delta\phi_o$ and ϕ_1^E), which is:

$$\left(\delta\phi_1^E\right)_{l=1} = -(B_\phi)_{l=1} \quad \text{on } \partial\Omega \text{ boundary.} \tag{42}$$

3 THE NUMERICAL APPROXIMATION

Since we are focused on the low frequency deformations (with respect to seismic frequencies), we can neglect the inertial term. The problem is therefore quasi-static, the elastic deformations mainly take place at the frequency of the external forces. For this reason, we find it is more convenient to work in the frequency domain. Let us define ω the angular frequency.

We solve the equations using a finite element method called the spectral element method. In such approach, coupling the solid and the liquid part is not easy (Komatitsch & Tromp 2002a; Chaljub *et al.* 2003; Chaljub & Valette 2004). Here, we develop our method using the equations for the solid only. We take into account the liquid part of the core by setting the rigidity parameter of the outer core very low. We show in the last section of the article that this approximation is valid with a good precision for mantle investigations.

The two types of problems considered in the previous sections require us to solve a coupled system of partial differential equations written for every point of Ω (a SNREI earth model) as:

$$-\rho_o\omega^2\,\vec{u} + \mathcal{A}\vec{u} + \rho_o\vec{\nabla}\phi = \vec{f}_1, \tag{43}$$

$$\mathcal{B}\phi + \vec{\nabla} \cdot (\rho_o\vec{u}) = \frac{1}{4\pi G}f_2, \tag{44}$$

with two differential operators. Here $\mathcal{A}$ is the classical elastodynamical operator defined by $\mathcal{A}\vec{u} = -\vec{\nabla} \cdot \overline{\overline{\mathbf{T}}}_1^L(\vec{u}) + \vec{\nabla}(\rho_o\vec{u} \cdot \vec{\nabla}\phi_o) - \vec{\nabla} \cdot (\rho_o\vec{u})\vec{\nabla}\phi_o$ and $\mathcal{B}$ is the operator defined by $\mathcal{B}\phi = \frac{1}{4\pi G}\Delta\phi$.

This system of equations describe the two problems previously presented, both unperturbed and perturbed. The right-hand side functions are reduced to $\vec{f}_1 = \rho_o\vec{f}_{\text{ext}}$ and $f_2 = 0$ in the unperturbed problem. They otherwise follow the relation (23) and (24) in the perturbed one. $\vec{u}$ and ϕ represent here $\vec{u}$ and ϕ_1^E or $\delta\vec{u}$ and $\delta\phi_1^E$ depending on the situation.

The system of equations is associated with either boundary conditions (10)–(16) or (25)–(31). We will used the notations with the 'B_q' functions introduced in the Section 2.2 (where q is any relevant subscript) in the both cases, considering that the B_q are null in the unperturbed problem (except for $\vec{B}_{\partial\Omega} = -p_{ext}\vec{n}_o$).

3.1 The variational formulations of the equations

In the next section, we use a finite element method to solve the system. The variational formulations of the equations are essential for this numerical approach. We present now the variational forms of the momentum equation and of the mass redistribution equation.

3.1.1 Momentum equation

Let us define an admissible displacement test function $\vec{v}$ expressed on Ω, which is continuous throughout the interfaces Σ. We multiply eq. (43) by $\vec{v}$ and we integrate the equation over the volume Ω:

$$\int_\Omega (-\rho_o\omega^2\vec{u} + \mathcal{A}\vec{u} + \rho_o\vec{\nabla}\phi) \cdot \vec{v}\, dV = \int_\Omega \vec{f}_1 \cdot \vec{v}\, dV. \tag{45}$$

Following the Green relation:

$$\int_\Omega (\vec{\nabla} \cdot \overline{\overline{\mathbf{T}}}_1^L(\vec{u})) \cdot \vec{v}\, dV = -\int_\Omega \overline{\overline{\mathbf{T}}}_1^L(\vec{u}) : \vec{\nabla}\vec{v}\, dV + \int_{\delta\Omega} (\vec{n}_o \cdot \overline{\overline{\mathbf{T}}}_1^L(\vec{u})) \cdot \vec{v}\, dV - \int_\Sigma [(\vec{n}_o \cdot \overline{\overline{\mathbf{T}}}_1^L(\vec{u})) \cdot \vec{v}]_-^+\, dV. \tag{46}$$

Since $\vec{v}$ is continuous throughout Σ, we can show, using the traction boundary conditions, that:

$$\int_\Omega (\vec{\nabla} \cdot \overline{\overline{\mathbf{T}}}_1^L(\vec{u})) \cdot \vec{v}\, dV = -\int_\Omega \overline{\overline{\mathbf{T}}}_1^L(\vec{u}) : \vec{\nabla}\vec{v}\, dV + \int_{\delta\Omega} \vec{B}_{\partial\Omega} \cdot \vec{v}\, dV - \int_{\Sigma_{SS}} \vec{B}_{\mathbf{T}}^S \cdot \vec{v}\, dV - \int_{\Sigma_{SF}} \vec{B}_{\mathbf{T}}^F \cdot \vec{v}\, dV, \tag{47}$$

Valette (1986) showed that for a SNREI earth model:

$$\int_\Omega (\vec{\nabla}(\rho_o\vec{u} \cdot \vec{\nabla}\phi_o) - \vec{\nabla} \cdot (\rho_o\vec{u})\vec{\nabla}\phi_o) \cdot \vec{v}\, dV = \int_\Omega \rho_o\, Sym\{(\vec{v} \cdot \vec{g}_o)\vec{\nabla} \cdot \vec{u} - \vec{u} \cdot \vec{\nabla}(\vec{v} \cdot \vec{g}_o)\}\, dV,$$

with $Sym\{f(u, v)\} = \frac{1}{2}(f(u, v) + f(v, u))$.

Following the relation (5), we define a symmetrical bilinear a form such as:

$$a(\vec{u}, \vec{v}) = \int_\Omega \lambda(\vec{\nabla} \cdot \vec{u})(\vec{\nabla} \cdot \vec{v})\, dV + \int_\Omega \frac{\mu}{2}(\vec{\nabla}\vec{u} + \vec{\nabla}\vec{u}^T) : (\vec{\nabla}\vec{v} + \vec{\nabla}\vec{v}^T)\, dV$$

$$+ \int_\Omega Sym\{(\vec{v} \cdot \vec{g})\vec{\nabla} \cdot \vec{u} - \vec{u} \cdot \vec{\nabla}(\vec{v} \cdot \vec{g})\}\, dV$$

The variational form of the momentum equation can then be simply expressed:

$$-\omega^2 \int_\Omega \vec{u} \cdot \vec{v}\, \rho_o\, dV + a(\vec{u}, \vec{v}) + \int_\Omega \rho_o\vec{\nabla}\phi \cdot \vec{v}\, dV = l(\vec{v}), \tag{48}$$

with the real linear form define on Ω:

$$l(\vec{v}) = \int_\Omega \vec{f}_1 \cdot \vec{v}\, dV - \int_{\delta\Omega} \vec{B}_{\partial\Omega} \cdot \vec{v}\, dV + \int_{\Sigma_{SS}} \vec{B}_{\mathbf{T}}^S \cdot \vec{v}\, dV + \int_{\Sigma_{SF}} \vec{B}_{\mathbf{T}}^F \cdot \vec{v}\, dV. \tag{49}$$

3.1.2 Mass redistribution equation

Following the same approach that for the momentum equation, we define an admissible potential test function ψ, which is continuous throughout the interfaces $\partial\Omega \cup \Sigma_\rho$. We multiply eq. (44) by ψ and we integrate the product over Ω:

$$\int_\Omega (\Delta\phi + 4\pi G\vec{\nabla} \cdot (\rho_o\vec{u}))\psi\, dV = \int_\Omega f_2 \psi\, dV. \tag{50}$$

Following the Green relation and using the boundary conditions, we have:

$$\int_\Omega (\Delta\phi + 4\pi G\vec{\nabla} \cdot (\rho_o\vec{u}))\psi\, dV = -\int_\Omega (\vec{\nabla}\phi + 4\pi G\rho_o\vec{u}) \cdot \vec{\nabla}\psi\, dV + \int_{\partial\Omega} \vec{n}_o \cdot \vec{\xi}(\vec{u}, \phi)\psi\, dS - \int_{\Sigma_\rho} [\vec{n}_o \cdot \vec{\xi}(\vec{u}, \phi)\psi]_-^+\, dS \tag{51}$$

$$= -\int_\Omega (\vec{\nabla}\phi + 4\pi G\rho_o\vec{u}) \cdot \vec{\nabla}\psi\, dV + \int_{\partial\Omega} (\vec{n}_o \cdot \vec{\nabla}\phi)^{ext}\psi\, dS - \int_{\partial\Omega\cup\Sigma_\rho} B_\xi \psi\, dS \tag{52}$$

Note that the term $\int_{\partial\Omega}(\vec{n}_o \cdot \vec{\nabla}\phi)^{ext} \psi\, dS$ is unknown. We know that ϕ follows the Laplace equation in the exterior of the planet and that it tends to zero when the distance of the planet tends to infinity. One can show classically that if we expand the gravitational potential in spherical harmonics:

$$\phi^{ext}(r, \theta, \varphi) = \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \left(\frac{a}{r}\right)^{l+1} \phi_{lm} \mathcal{Y}_l^m(\theta, \varphi), \tag{53}$$

where $\mathcal{Y}_l^m$ is the complex spherical harmonic of degree l and order m and ϕ_{lm} is a spherical harmonic coefficient (a is the SNREI earth radius). Using this information, Chaljub & Valette (2004) showed that the previous integration can be expressed as:

$$\int_{\partial\Omega} (\vec{n}_o \cdot \vec{\nabla}\phi)^{\text{ext}} \psi \, dS = -\sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \frac{l+1}{a^3} \int_{\partial\Omega} \phi \, \bar{\mathcal{Y}}_l^m \, dS \int_{\partial\Omega} \psi \, \mathcal{Y}_l^m \, dS, \tag{54}$$

with $\bar{\mathcal{Y}}_l^m$ the complex conjugate of $\mathcal{Y}_l^m$. Note that ϕ here refers to ϕ^{int} the potential in the interior of the planet.

We define the symmetrical bilinear form b:

$$b(\phi, \psi) = \frac{1}{4\pi G} \left[\int_\Omega \vec{\nabla}\phi \cdot \vec{\nabla}\psi \, dV + \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \frac{l+1}{a^3} \int_{\partial\Omega} \phi \, \bar{\mathcal{Y}}_l^m \, dS \int_{\partial\Omega} \psi \, \mathcal{Y}_l^m \, dS \right]. \tag{55}$$

The variational form of the mass redistribution equation can be simply expressed:

$$b(\phi, \psi) + \int_\Omega \rho_o \vec{u} \cdot \vec{\nabla}\psi \, dV = k(\psi), \tag{56}$$

with k a real linear form define on Ω:

$$k(\psi) = -\int_\Omega f_2 \, \psi \, dV + \int_{\partial\Omega\cup\Sigma_\rho} B_\xi \, \psi \, dS. \tag{57}$$

3.1.3 Global system and the perturbed particular case

The problem we will have to solve using a finite element method is, for all admissible $\vec{v}$ function and ψ function find $\vec{u}$ and ϕ solutions of the equations:

$$-\omega^2 \int_\Omega \vec{u} \cdot \vec{v} \, \rho_o \, dV + a(\vec{u}, \vec{v}) + \int_\Omega \rho_o \vec{\nabla}\phi \cdot \vec{v} \, dV = l(\vec{v}), \tag{58}$$

$$b(\phi, \psi) + \int_\Omega \rho_o \vec{u} \cdot \vec{\nabla}\psi \, dV = k(\psi). \tag{59}$$

One can show that for a reasonable frequency this problem presents a unique solution $\vec{u}$ and ϕ which, in the unperturbed case, corresponds to the solution of the elastogravitational eqs (8) and (9) (see Valette 1986). Naturally $\vec{u}$ and ϕ have the same attributes than $\vec{v}$ and ψ, they are continuous through each boundary interface. The problem is more complicated for the perturbed case because the solutions have to follow the boundary conditions (26) and (30) and not the homogeneous conditions. In order to take into account this aspect of the problem, we have to make a change of variables in the perturbed case. We denote by $\vec{u}_o$ the displacement of particles within the planet between the reference SNREI earth Ω and the perturbed reference earth Ω'. We define two new variables $\vec{w}$ and φ such as (using the notation defined in Section 2):

$$\vec{w} = \delta\vec{u} + \vec{u}_o \cdot \vec{e}_r \, \partial_r \vec{u}, \tag{60}$$

$$\varphi = \delta\phi_1^E + \vec{u}_o \cdot \vec{e}_r \, \partial_r \phi_1^E, \tag{61}$$

with $\vec{e}_r$ the unit radial vector and ∂_r the radial derivative. Following the boundary relations, $\vec{w}$ and φ are continuous through the internal boundaries $\partial\Omega \cup \Sigma \cup \Sigma_\rho$.

$$-\omega^2 \int_\Omega \vec{w} \cdot \vec{v} \, \rho_o \, dV + a(\vec{w}, \vec{v}) + \int_\Omega \rho_o \vec{\nabla}\varphi \cdot \vec{v} \, dV = l'(\vec{v}), \tag{62}$$

$$b(\varphi, \psi) + \int_\Omega \rho_o \vec{w} \cdot \vec{\nabla}\psi \, dV = k'(\psi), \tag{63}$$

with, if we denote $C_u = \vec{u}_o \cdot \vec{e}_r \, \partial_r \vec{u}$ and $C_\phi = \vec{u}_o \cdot \vec{e}_r \, \partial_r \phi_1^E$:

$$l'(\vec{v}) = l(\vec{v}) + \omega^2 \int_\Omega \vec{C}_u \cdot \vec{v} \, \rho_o \, dV - a(\vec{C}_u, \vec{v}) - \int_\Omega \rho_o \vec{\nabla}C_\phi \cdot \vec{v} \, dV, \tag{64}$$

$$k'(\psi) = k(\psi) - b(C_\phi, \psi) - \int_\Omega \rho_o \vec{C}_u \cdot \vec{\nabla}\psi \, dV. \tag{65}$$

The new variational forms (62) and (63) are similar to (58) and (59) and have the same properties. By solving this system we find the variables $\vec{w}$ and φ which are continuous through the internal boundaries. We can then calculate the solutions $\delta\vec{u}$ and $\delta\phi$ from the relations (60) and (61).

3.2 The spectral element method approximation

3.2.1 The grid

The reference planet Ω has to be first discretized. We use the mesh developed by Chaljub *et al.* (2003) called the 'cubed sphere mesh'. It has been successfully used in many spectral element applications for seismology (see Komatitsch & Tromp 2002b; Capdeville *et al.* 2003;
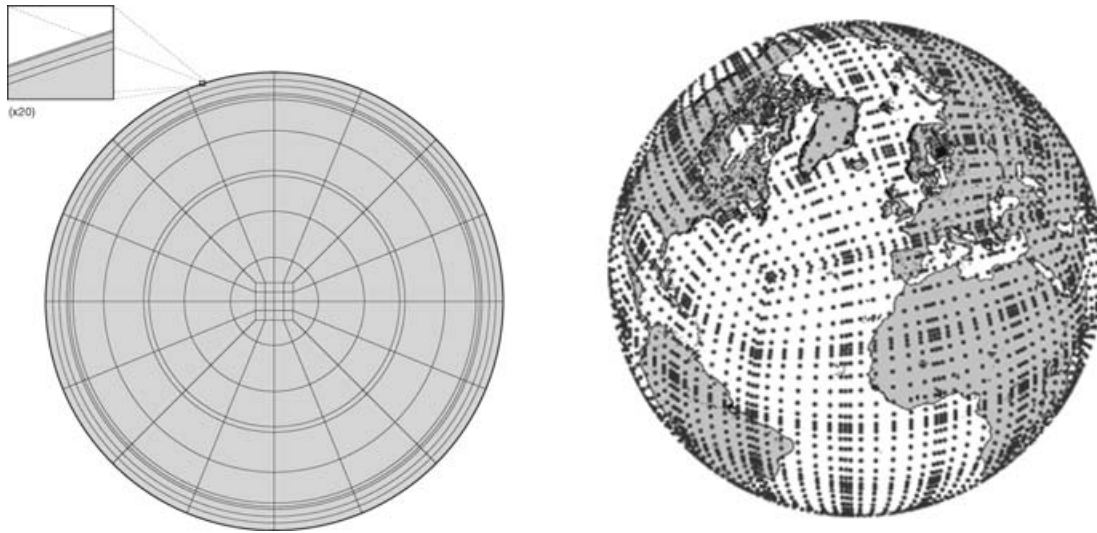
Figure 2. An example of grid used for the spectral element method. The left and right panels show the distribution of volume elements in a transversal section of PREM earth model (each 3-D element contains about 1000 points) and the surface points of the grid, respectively.

Chaljub & Valette 2004). The mesh composition is based on a Ronchi *et al.* (1996) transformation, where a sphere is divided into six lateral regions. The advantages of the mesh are that it is mostly homogeneous and that it is easily divided in hexahedral volume elements, which is needed for the spectral element approximation. The regions of the mesh are divided in volume elements, each one containing a certain number of points of the grid. The hexahedral elements are chosen to be conformed, that is, each face of each element is stuck to one and only one other element. In the centre, the six lateral regions are joined by a seventh cubic region, following the Chaljub *et al.* (2003) transformation (see Fig. 2).

3.2.2 Variational form approximations

The spectral element method is a finite element method based on Lagrange polynomial interpolation and on sum approximation using Gauss–Lobatto–Legendre points (classically linked to spectral collocation method). In Appendix B, we present a summary of spectral element variational form discretization. We do not present the details of the method here since it has been previously used by various authors (see particularly Komatitsch & Tromp 2002a; Chaljub *et al.* 2003).

3.2.3 The global linear system and its solution

Using spectral element approximation scheme, the different variational forms (58) and (59) [or equivalently (62) and (63)] can be reduced to linear systems of equations. One can show that a simple term of the variational forms would be expressed (see appendix B for more details):

$$\int_{\Omega} \phi \, \psi \, dV \approx \Psi \cdot \mathbf{M} \cdot \Phi, \tag{66}$$

where Φ and Ψ are the vectors whose each component represents the value of ϕ and ψ in a point of the grid. $\mathbf{M}$ is a matrix associated with the different mesh parameters and earth parameters. Following this principle, the different terms of variational forms can be expressed:

$$\int_{\Omega} \vec{u} \cdot \vec{v} \, \rho_o \, dV \approx \mathbf{V} \cdot \mathbf{M} \cdot \mathbf{U}, \tag{67}$$

$$a(\vec{u}, \vec{v}) \approx \mathbf{V} \cdot \mathbf{K} \cdot \mathbf{U}, \tag{68}$$

$$\int_{\Omega} \vec{\nabla}\phi \cdot \vec{v} \, \rho_o \, dV \approx \mathbf{V} \cdot \mathbf{N}^T \cdot \Phi, \tag{69}$$

$$\int_{\Omega} \vec{u} \cdot \vec{\nabla}\psi \, \rho_o \, dV \approx \Psi \cdot \mathbf{N} \cdot \mathbf{U}, \tag{70}$$

$$b(\phi, \psi) \approx \Psi \cdot \mathbf{L} \cdot \Phi, \tag{71}$$

$$h(\vec{v}) \approx \mathbf{V} \cdot \mathbf{F}_1, \tag{72}$$

$$k(\psi) \approx \Psi \cdot \mathbf{F}_2, \tag{73}$$

where $\mathbf{N}^T$ is the transposed matrix of $\mathbf{N}$. Since the formulation is valid for every $\mathbf{V}$ and every $\mathbf{\Psi}$, we can divide the equations respectively by $\mathbf{V}$ and $\mathbf{\Psi}$. The approximation of the system of equations is reduced to two linear systems:

$$(-\omega^2 \mathbf{M} + \mathbf{K}) \cdot \mathbf{U} + \mathbf{N}^T \cdot \Phi = \mathbf{F}_1, \tag{74}$$

$$\mathbf{N} \cdot \mathbf{U} + \mathbf{L} \cdot \Phi = \mathbf{F}_2. \tag{75}$$

We obtain a large linear system composed of four matrix $\mathbf{K}$, $\mathbf{L}$, $\mathbf{M}$ and $\mathbf{N}$. The first three are symmetric, the global matrix $\mathbf{A}$ is therefore symmetric:

$$\mathbf{A} = \begin{bmatrix} -\omega^2 \mathbf{M} + \mathbf{K} & \mathbf{N}^T \\ \mathbf{N} & \mathbf{L} \end{bmatrix}. \tag{76}$$

    The system is not simple to solve because one can show that the matrix $\mathbf{A}$ is not purely positive definite (indefinite matrix). Consequently, most of the classical methods such as conjugate gradient are not theoretically adapted to solve this linear system.

    The planet response presents several components. The major part of the response corresponds to global deformation with volume variation. However, a part of the planet response corresponds to pure rotations, the degree 1 toroidal part. It is well known that, in static, the elastogravitational equations have no solutions for these rotational modes of deformations. In fact, we can show that the matrix $\mathbf{K}$ is singular due to this type of deformations. We see here how it is important to solve the equations in dynamics, since $-\omega^2 \mathbf{M} + \mathbf{K}$ is not singular (for $\omega \neq 0$). The matrix is nevertheless indefinite due to this degree 1 toroidal modes, and consequently so is $\mathbf{A}$. In order to solve the linear system we investigate an iterative method that can take into account the indefiniteness of the matrix and therefore rotational modes of deformation. We use a classical method based on *Krylov subspace theory* known as SYMMLQ and developed by Paige & Saunders (1975). The method is efficient for this type of problem and has been largely used in numerical studies.

    Note that in many cases the system can be considered as positive definite. If the second member does not present degree 1 toroidal component, we can show that with iterative methods, the system acts as if it were positive definite. In this case, it is well adapted to conjugate gradient method. Métivier *et al.* (2005) used this method in a previous paper in order to investigate static tide deformations with no degree 1 toroidal modes.

    The code is implemented on a multi-processor computer. The CPU time is variable depending naturally on the grid. One of the problems is that the iterative method of solution is presently not well preconditioned (it is presently preconditioned with the diagonal of the global matrix). Therefore, the time of convergence can be variable depending on the type of earth we investigate. Typically on eight processors (ES40 processor of 500 Mz each at the Institut de Physique du Globe, Département de Modélisation Physique et Numérique), the program runs during a few minutes to a few hours.

4 VALIDATION

In order to validate our numerical scheme, we have run two test cases. In a previous publication Métivier *et al.* (2005) presented a first validation of the model on tide investigations without detailing the numerical model. Here we focus on the Earth's response to surface loadings. We first determine the load Love numbers for a radially stratified earth (PREM earth model, see Dziewonski & Anderson (1981)) in order to compare them with a previous calculation made by Wahr *et al.* (1998). This calculation also permits to investigate the impact of the external core approximation we use. Then, we study the perturbation due to ellipticity on load Love numbers for a homogeneous incompressible earth. We then compare our solutions with analytical ones.

4.1 Surface loadings

The Earth is deformed under the weight of the external fluid layers like the oceans or the atmosphere. The fluid layers are classically modelled as thin surface layers, we denote by σ_{ext} their surface density. The mass of a layer induces a forcing potential at each point within Ω:

$$V_{\text{ext}}(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} V_{lm} \left(\frac{r}{a} \right)^l \mathcal{Y}_l^m(\theta, \varphi). \tag{77}$$

The mass induces a pressure on the Earth's surface linked with the layer density:

$$p_{\text{ext}}(\theta, \varphi) = -\sigma_{\text{ext}} \, \vec{g}_o \cdot \vec{n}_o, \tag{78}$$

with $\vec{g}_o$ the reference gravity of the spherical earth, and $\vec{n}_o = \vec{e}_r$ the unit radial vector. The surface pressure can be therefore linked to the potential (Wahr *et al.* 1998):

$$p_{\text{ext}}(\theta, \varphi) = -\sum_{l=0}^{\infty} \sum_{m=-l}^{+l} V_{lm} \frac{2l+1}{3} \, \tilde{\rho} \, V_{lm} \mathcal{Y}_l^m(\theta, \varphi), \tag{79}$$

with $\tilde{\rho}$ the mean density of the planet so that the reference gravity on the Earth's surface is $\vec{g}_o(r = a) = -\frac{4}{3}\pi G \tilde{\rho} a \vec{e}_r$.

Table 1. The first 30 load Love numbers calculated with our numerical model and the k'_l calculated by Wahr *et al.* (1998).

	Our solution			Wahr solution	Absolute deviation
l	h'_l	l'_l	k'_l	k'_l	Δ
1	−1.286	−0.896	−1.000	–	–
2	−0.989	0.023	−0.304	−0.303	0.001
3	−1.050	0.069	−0.195	−0.194	0.001
4	−1.053	0.059	−0.134	−0.132	0.002
5	−1.087	0.046	−0.105	−0.104	0.001
6	−1.145	0.039	−0.090	−0.089	0.001
7	−1.214	0.034	−0.082	−0.081	0.001
8	−1.285	0.032	−0.077	−0.076	0.001
9	−1.356	0.030	−0.073	−0.072	0.001
10	−1.425	0.028	−0.069	−0.069	$<10^{-3}$
11	−1.491	0.027	−0.066	−0.067	0.001
12	−1.554	0.026	−0.064	−0.064	$<10^{-3}$
13	−1.614	0.026	−0.062	−0.062	$<10^{-3}$
14	−1.671	0.025	−0.060	−0.060	$<10^{-3}$
15	−1.726	0.025	−0.058	−0.058	$<10^{-3}$
16	−1.779	0.025	−0.057	−0.057	$<10^{-3}$
17	−1.829	0.024	−0.055	−0.055	$<10^{-3}$
18	−1.876	0.024	−0.054	−0.054	$<10^{-3}$
19	−1.921	0.024	−0.052	−0.052	$<10^{-3}$
20	−1.964	0.023	−0.051	−0.051	$<10^{-3}$
21	−2.005	0.023	−0.050	−0.050	$<10^{-3}$
22	−2.044	0.023	−0.049	−0.049	$<10^{-3}$
23	−2.081	0.023	−0.048	−0.048	$<10^{-3}$
24	−2.116	0.022	−0.047	−0.047	$<10^{-3}$
25	−2.149	0.022	−0.045	−0.046	0.001
26	−2.180	0.022	−0.044	−0.044	$<10^{-3}$
27	−2.210	0.021	−0.043	−0.043	$<10^{-3}$
28	−2.239	0.021	−0.043	−0.042	0.001
29	−2.266	0.021	−0.042	−0.041	0.001
30	−2.291	0.021	−0.041	−0.040	0.001

4.2 Validity of external core approximation

In our calculation the liquid core is modelled as an elastic solid with a small rigidity (about 10^9 Pa). In order to validate this approximation we investigate the deformation induced by surface loadings on PREM earth model. We compare our solution with the solutions proposed by Wahr *et al.* (1998). We apply an arbitrary loading map on the surface of the Earth. We calculated the response of the planet using a static assumption ($\omega = 0$). In this calculation, no degree one toroidal deformation is possible since the external forcing is a load on the spherical surface. The static approximation is therefore valid. Using numerical Legendre transform, we extract spherical harmonic coefficients of the surface displacement and the potential perturbation of the Earth. We then calculate the so-called degree l loading Love numbers h'_l, l'_l, k'_l of the planet defined such as:

$$\vec{u}(r = a) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{V_{lm}}{g_o} \left(h'_l \, \mathcal{Y}_{lm} \vec{e}_r + l'_l \, a \, \vec{\nabla}_\Sigma \mathcal{Y}_{lm} \right), \tag{80}$$

$$\phi_1^E(r = a) = - \sum_{l=0}^{\infty} \sum_{m=-l}^{l} V_{lm} \, k'_l \, \mathcal{Y}_{lm}. \tag{81}$$

We present the Love numbers calculated for degrees 1 to 30 in Table 1. We compare our k'_l with the one calculated by Wahr *et al.* (1998) (the h'_l and l'_l are not given by the author).

The absolute deviation between the k'_l Love numbers is also presented in Table 1. Wahr *et al.* (1998) gave solutions to the first three digits after the comma. We present here our solution with the same number of digits after the comma. For l degrees from 12 to 24 the differences between the two solutions are smaller than the precision of the solutions. For low l degrees and for high l degrees, the difference is about 0.001. It is not the exact difference but the maximum bound of this difference since the solutions are rounded. This worse agreement between the solutions is due to two types of approximation. For low l degree, it is due to the external core approximations because only the low degree components of the loading function can reach the core. For high degree it is due to the mesh which is not very dense in the surface of the planet for this test (see Fig. 2). Nevertheless, the agreement is generally good. The mesh can be densified and we see that the external core approximation have a relatively small impact on the solution (inferior to 0.3 per cent). This relative error has been calculated in the unperturbed case with a SNREI earth model. Since we use the same solution method for the perturbed case, the relative error induced by the

core approximation in the perturbed case would be of the same order of magnitude than in the unperturbed case. The relative error related to this approximation is consequently negligible for the type of problem we are looking for.

4.3 Impact of ellipticity on loading deformation

We focus now on the impact of hydrostatic ellipticity on the response of the Earth to surface loadings. Under the action of the centrifugal potential, the Earth is flattened following a hydrostatic equilibrium. The shape of the planet consequently slightly differs from spherical earth and then the vertical gravity is modified.

The shape of the hydrostatic ellipsoidal planet follows an equipotential surface (Clairaut's equation), and consequently the perturbation of gravity potential and surface topography are linked. The surface topography is:

$$h = -\frac{2}{3}\alpha\, a\, \frac{3\cos^2\theta - 1}{2}, \quad \text{with } \alpha \text{ the hydrostatic flattening.} \tag{82}$$

The gravity on the ellipsoïd can be written:

$$\vec{g}_o + \delta\vec{g}_o = (g_o + \delta g_o)\,\vec{n}_e, \quad \text{with } \vec{n}_e = \vec{e}_r - \alpha\sin(2\theta)\,\vec{e}_\theta. \tag{83}$$

For a homogeneous earth, we have $g_o = -\frac{4}{3}\pi G\rho_o a$, with ρ_o constant, and $\delta g_o = -\frac{h}{a}g_o$.

The total pressure applied on the ellipsoidal Earth surface is:

$$(p_{\text{ext}} + \delta p_{\text{ext}})\,\vec{n}_e = -\,\sigma_{\text{ext}}\,(g_o + \delta g_o)\,\vec{n}_e. \tag{84}$$

The perturbation of surface pressure is:

$$\delta p_{\text{ext}} = -p_{\text{ext}}\frac{h}{a}. \tag{85}$$

It has been known for one century that elastogravitational deformation problems have analytical solutions for a homogeneous incompressible planet. Love (1911) developed this analytical solution to determine the perturbation of ellipticity on semi diurnal tides. We retrieved such solution and developed it with a Lagrangian formalism in order to validate the solution of our numerical model (Greff-Lefftz *et al.* 2005). The comparison between the analytical and the numerical solutions are presented in Métivier *et al.* (2005). Here we present new validation results involving surface loading deformations. We investigate the impact of hydrostatical ellipticity on the Earth surface loading response. Because the first observation of loading impact is the temporal variations of J_n coefficients, we focus our work on zonal variation of k' Love number. Moreover, the coupling between ellipticity and zonal surface loadings cannot generate degree one toroidal modes. It permits to solve the problem in the static approximation, which is needed to compare with the analytical solution.

Following the approach of Greff-Lefftz *et al.* (2005), we find the analytical solution of this problem. Let us define the parameter $\bar{\mu}_l = \frac{2l^2+4l+3}{l}\frac{\mu}{\rho_o g_o a}$, where μ is the rigidity, which depends upon the spherical harmonic degree l.

For a zonal loading of degree l, the coupling with ellipticity generates perturbations of degree $l-2$, l and $l+2$. We denote by k'_l the classical Love numbers of degree l and by Δk_l^-, Δk_l^o and Δk_l^+ the perturbation of this Love number on the ellipsoidal surface. The external potential variation (known as free space potential) has the form:

$$\left(\phi_1^E + \delta\phi_1^E\right)^{\text{free space}} = V_l\left[\Delta k_l^-\left(\frac{a}{r}\right)^{l-1}\mathcal{Y}_{l-2}^0 + \left(k_l + \Delta k_l^0\right)\left(\frac{a}{r}\right)^{l+1}\mathcal{Y}_l^0 + \Delta k_l^+\left(\frac{a}{r}\right)^{l+3}\mathcal{Y}_{l+2}^0\right]. \tag{86}$$

The analytical solution of these Love numbers are presented in Table 2 for the degree 1 to 5. The Table 3 presents the comparison between numerical and analytic solutions of Love numbers. The values of parameter used for the calculation are $\rho_o = 5520$ kg m^{-3}, $\mu = 1.15 \times 10^{11}$ Pa, $\alpha = 1/232$ and $a = 6371$ km. In our numerical model, the incompressible Earth is modelled as a compressible Earth with high compressibility modulus (typically about 10^{14}).

Table 2. Analytical solution of k' Love number and of the Love number perturbations. α is the earth flattening and $\bar{\mu}_l$ the effective rigidity of degree l (see the text for the definition).

l	k'_l	Δk_l^-	Δk_l^0	Δk_l^+
1	-1	$-$	0	$\frac{3\alpha}{5}\frac{(9+22\,\bar{\mu}_1)}{11\,\bar{\mu}_1+9}$
2	$\frac{-1}{1+\bar{\mu}_2}$	0	$-\frac{2\alpha}{1995}\frac{(-475+221\,\bar{\mu}_2)}{(\bar{\mu}_2+1)^2}$	$\frac{4\alpha}{105}\frac{(855+1202\,\bar{\mu}_2)}{(\bar{\mu}_2+1)(38+51\,\bar{\mu}_2)}$
3	$\frac{-1}{1+\bar{\mu}_3}$	0	$\frac{4\alpha}{3465}\frac{(539+236\,\bar{\mu}_3)}{(\bar{\mu}_3+1)^2}$	$\frac{10\alpha}{693}\frac{(4235+5693\,\bar{\mu}_3)}{(\bar{\mu}_3+1)(55+73\,\bar{\mu}_3)}$
4	$\frac{-1}{1+\bar{\mu}_4}$	$\frac{34\alpha}{35}\frac{(45+17\,\bar{\mu}_4)}{(\bar{\mu}_4+1)(51+38\,\bar{\mu}_4)}$	$\frac{60\alpha}{1309}\frac{(17+12\,\bar{\mu}_4)}{(\bar{\mu}_4+1)^2}$	$\frac{\alpha}{143}\frac{(3315+4297\,\bar{\mu}_4)}{(17+22\,\bar{\mu}_4)(\bar{\mu}_4+1)}$
5	$\frac{-1}{1+\bar{\mu}_5}$	$\frac{730\alpha}{693}\frac{(41\,\bar{\mu}_5+77)}{(\bar{\mu}_5+1)(55\,\bar{\mu}_5+73)}$	$\frac{\alpha}{93951}\frac{(72541\,\bar{\mu}_5+88330)}{(\bar{\mu}_5+1)^2}$	$\frac{7\alpha}{143}\frac{(21235\,\bar{\mu}_5+16863)}{(\bar{\mu}_5+1)(645\,\bar{\mu}_5+511)}$

Table 3. Comparison between analytical and numerical Love numbers for a homogeneous incompressible earth, with $\rho_o = 5520$ kg·m^{-3}, $\mu = 1.15 \, 10^{11}$ Pa, $\alpha = 1/232$ and $a = 6371$ km.

l	Love number	The analytical solution	The numerical solution	Relative error
1	k'_1	-1	-1	–
	Δk_1^-	0	0	–
	Δk_1^0	0	0	–
	Δk_1^+	4.61741×10^{-3}	4.61768×10^{-3}	0.006 per cent
2	k'_2	-0.24034	-0.24043	0.04 per cent
	Δk_2^-	0	0	–
	Δk_2^0	-0.05579×10^{-3}	-0.05566×10^{-3}	0.23 per cent
	Δk_2^+	0.92210×10^{-3}	0.92458×10^{-3}	0.25 per cent
3	k'_3	-0.21460	-0.21467	0.03 per cent
	Δk_3^-	0	0	–
	Δk_3^0	0.32145×10^{-3}	0.32159×10^{-3}	0.04 per cent
	Δk_3^+	1.03871×10^{-3}	1.04013×10^{-3}	0.14 per cent
4	k'_4	-0.19077	-0.19081	0.02 per cent
	Δk_4^-	0.44086×10^{-3}	0.44110×10^{-3}	0.05 per cent
	Δk_4^0	0.48823×10^{-3}	0.48848×10^{-3}	0.05 per cent
	Δk_4^+	1.12283×10^{-3}	1.12232×10^{-3}	0.05 per cent
5	k'_5	-0.17072	-0.17075	0.01 per cent
	Δk_5^-	0.62930×10^{-3}	0.63083×10^{-3}	0.24 per cent
	Δk_5^0	0.58928×10^{-3}	0.59054×10^{-3}	0.20 per cent
	Δk_5^+	1.18630×10^{-3}	1.18744×10^{-3}	0.09 per cent

We present the relative error of our solution in the Table 3. It is generally inferior to 0.2 per cent for all the loading Love numbers. This error is principally due to our approximation of incompressibility. In the present application, this precision on Love numbers corresponds approximately to a precision of 1 μm and of 0.1 nanoGal on displacement and on gravity.

5 CONCLUSION

With our new model, we are able to calculate the low frequency elastogravitational deformations of a planet with a realistic 3-D internal structure. Using first-order perturbation theory, the model takes into account lateral variations of density and of elastic parameters, topography on the interfaces and deviation to hydrostatic pre-stress state. The model has been validated with simple applications against theoretical and analytical results. We investigate the Earth's response to surface loadings and the impact of hydrostatic ellipticity on Earth deformations. We show that the elastic approximation of external core is good enough for the type of problem we are interested in. For the Earth, the relative error introduced by this approximation is at maximum 0.3 per cent. The experiment involving the hydrostatic ellipsoid on Earth loading deformation shows that the numerical model can be very precise. In our application, the precision is about 1 μm in the displacement and 0.1 nanoGal in the gravity.

We note that the computing time can be quite important depending of the type of earth model we use. However, in the major experiments we did presently, the computation of the solution took less than 5 hr. That can be improved further using a more suitable preconditioning method.

Different geophysical problems can be investigated with this model. We will be able for the first time to determine what exactly the body tide perturbations are induced by mantle convection. We address some aspects of this problem in Métivier *et al.* (2006) where we modelled plumes and superplumes in the mantle as spherical density heterogeneities. Other investigations would be interesting, for example, to calculate the impact of the crustal dichotomy on the loading response of the Earth. Moreover, this type of problem is easily adapted for other telluric planets, for example Mars, which is probably far from a hydrostatic equilibrium state.

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REFERENCES

Capdeville, Y., Chaljub, E., Vilotte, J.-P., & Montagnier, J.-P., 2003. Coupling the spectral element method with a modal solution for elastic wave propagation in global earth models, *Geophys. J. Int.,* **152,** 34–67.

Chaljub, E. & Valette, B., 2004. Spectral element modeling of three dimensional wave propagation in a self-gravitating Earth with an arbitrarily stratified outer core, *Geophys. J. Int.,* **158,** 131–141.

Chaljub, E., Capdeville, Y. & Vilotte, J.-P., 2003. Solving elastodynamics in a fluid-solid heterogeneous sphere: a parallel spectral element approximation on non-conforming grids, *J. Comput. Phys.,* **187,** 457–491.

Courtillot, V., Davaille, A., Besse, J. & Stock, J., 2003. Three distinct types of hotspots in earth's mantle, *Earth planet. Sci. Lett.,* **205,** 295–308.

Dahlen, F.A., 1972. Elastic dislocation theory for a self-gravitating elastic configuration with an initial static stress field, *Geophys. J. R. astr. Soc.,* **28,** 357–383.

Dahlen, F.A. & Tromp, J., 1998. *Theoretical Global Seismology*, Princeton University Press, New Jersey.

Davaille, A., 1999. Simultaneous generation of hotspots and superswells by convection in a heterogeneous planetary mantle, *Nature,* **402,** 756–760.

Defraigne, P., 1997. Geophysical model of dynamical flattening of the Earth in agreement with the precession constant, *Geophys. J. Int.,* **130,** 47–56.

Defraigne, P., Dehant, V. & Wahr, J., 1996. Internal loading of an homogeneous compressible earth with phase boundaries, *Geophys. J. Int.,* **125,** 173–192.

Dehant, V., 1987. Tidal parameters for an inelastic Earth, *Phys. Earth Planet. Inter.,* **49,** 97–116.

Dehant, V., Defraigne, P. & Wahr, J.M., 1999. Tides for a convective Earth, *J. geophys. Res.,* **104,** 1035–1058.

Dziewonski, A.M. & Anderson, D.L., 1981. Preliminary Referential Earth Model, *Phys. Earth planet. Inter.,* **25,** 297–356.

Farrell, W.E., 1972. Deformation of the Earth by surface loads, *Rev. of Geophys. and Space Phys.,* **10,** 761–797.

Forte, A.M. & Mitrovica, J.X., 2001. Deep-mantle high-viscosity flow and thermochemical structure inferred from seismic and geodynamic data, *Nature,* **410,** 1049–1056.

Greff-Lefftz, M. & Legros, H., 1997. Some remarks about the degree one deformations of the Earth, *Geophys. J. Int.,* **131,** 699–723.

Greff-Lefftz, M., Métivier, L. & Legros, H., 2005. Analytical solutions of Love numbers for an hydrostatic ellipsoidal incompressible homogeneous Earth, using Lagrangian formulation, *Celest. Mech. and Dyn. Astron.,* **93,** 113–146.

Gu, Y.J., Dziewonski, A.M., Su, W.J & Ekström, G., 2001. Models of the mantle shear velocity and discontinuities in the pattern of lateral heterogeneities, *J. geophys. Res.,* **106,** 11 169–11 199.

Komatitsch, D. & Tromp, J., 2002a. Spectral-element simulations of global seismic wave propagation - I. Validation, *Geophys. J. Int.,* **149,** 390–412.

Komatitsch, D. & Tromp, J., 2002b. Spectral-element simulations of global seismic wave propagation - II. Three-dimensional models, oceans, rotation and self-gravitation, *Geophys. J. Int.,* **150,** 303–318.

Latytchev, K., Mitrovica, J.X., Tamisiea, M.E., Tromp, J., Christara, C.C. & Moucha, R., 2005a. GIA-induced secular variations in the Earth's long wavelength gravity field: influence of 3D-viscosity variations, *Earth planet. Sci. Lett.,* **240,** 322–327.

Latytchev, K., Mitrovica, J.X., Tromp, J., Tamisiea, M.E., Komatitsch, D. & Christara, C.C., 2005b. Glacial isostatic adjustment on 3-D Earth models: a finite volume formulation, *Geophys. J. Int.,* **161,** 421–444.

Lognonné, P. & Romanowicz, B., 1990. Modelling of coupled normal modes of the Earth: the spectral method, *Geophys. J. Int.,* **102,** 365–395.

Love, A.E.H., 1911. *Some problems of geodynamics*, Dover Publications, New York.

Métivier, L., Greff-Lefftz, M. & Diament, M., 2005. A new approach to compute accurate gravity time variations for a realistic Earth model with lateral heterogeneities, *Geophys. J. Int.,* **162,** 570–574.

Métivier, L., Greff-Lefftz, M. & Diament, M., 2006. Mantle lateral variations and elastogravitational deformations – II. Possible effects of a superplume on body tides, Geophys. J. Int., submitted.

Molodenskiy, S.M., 1977. The influence of horizontal inhomogeneities in the mantle on the amplitude of the tidal oscillations, *Izvestiya, Earth Physics,* **13,** 77–80.

Munk, W.H. & MacDonald, G.J.F., 1960. *The rotation of the Earth*, Cambridge University Press, London, 323pp.

Paige, C.C. & Saunders, M.A., 1975. Solution of sparse indefinite systems of linear equations, *SIAM. J. Numer. Anal.,* **12,** 617–629.

Ronchi, C., Iacono, R. & Paolucci, P.S., 1996. The "Cubed Sphere": a new method for the solution of partial differential equations in spherical geometry, *J. Comput. Phys.,* **124,** 93–114.

Smith, M.L., 1974. The scalar equations of infinitesimal elastic-gravitational motion for a rotating, slightly elliptical Earth, *Geophys. J. R. astr. Soc.,* **37,** 491–526.

Takeuchi, H., 1950. On the Earth tides of the compressible Earth of variable density and elasticity, *Trans. Amer. Geophys. Union,* **31,** No. 5, 651–689.

Thomson, S.W., 1862. Dynamical Problems regarding Elastic Spheroidal Shells and Spheroids of Incompressible Liquid, *Phil. Trans. R. Soc. Lond.,* **153,** 583–608.

Valette, B., 1986. About the influence of pre-stress upon adiabatic perturbations of the Earth, *Geophys. J. R. astr. Soc.,* **85,** 179–208.

Wahr, J., Molenaar, M. & Bryan, F., 1998. Time variability of Earth's gravity field: hydrological and oceanic effects and their possible detection using GRACE, *J. geophys. Res.,* **103,** 30 205–30 229.

Wahr, J.M., 1981. Body tides on an elliptical, rotating, elastic and oceanless Earth, *Geophys. J. R. astr. Soc.,* **64,** 677–703.

Wahr, J.M. & Bergen, Z., 1986. The effects of mantle anelasticity on nutations, Earth tides, and tidal variations in rotation rate, *Geophys. J. R. astr. Soc.,* **64,** 633–668.

Wang, R., 1991. Tidal deformations of a rotating, spherically asymmetric, visco-elastic and laterally heterogeneous Earth, *PhD thesis*, Univ. of Kiel, Kiel, Germany.

Woodhouse, J.H. & Dahlen, F.A., 1978. The effect of a general aspherical perturbation on the free oscillations of the Earth, *Geophys. J. R. astr. Soc.,* **53,** 335–354.

Zhong, S., Paulson, A. & Wahr, J., 2003. Three-dimensional finite element modeling of Earth's viscoelastic deformations: effects of lateral variations in lithospheric thickness, *Geophys. J. Int.,* **155,** 679–695.

APPENDIX A: B_q CALCULATION

The complexity of the formulation of B_q terms arise from the fact that the boundary surfaces present a topography h and a normal perturbation $-\vec{\nabla}_\Sigma h$. Suppose any tensorial variable **q** defined on $\vec{x} \in \Omega$. In the gravitoelasticity problem, we have two types of limit conditions, one classically involving the continuity of the tensor, the other involving its normal continuity. In the unperturbed problem the two types of limit condition are:

$$[\mathbf{q}]_-^+ = 0 \qquad \forall \vec{x} \in \Sigma, \tag{A1}$$

$$[\vec{n}_o \cdot \mathbf{q}]_-^+ = 0 \qquad \forall \vec{x} \in \Sigma. \tag{A2}$$

In the perturbed problem the two types of limit condition are expressed on the new boundaries Σ', as follows:

$$[\mathbf{q} + \delta\mathbf{q}]_-^+ = 0 \qquad \forall\, \vec{x}' = (\vec{x} + h\,\vec{n}_o) \in \Sigma', \tag{A3}$$

$$[\vec{n}_o \cdot \mathbf{q} + \delta(\vec{n}_o \cdot \mathbf{q})]_-^+ = 0 \qquad \forall\, \vec{x}' = (\vec{x} + h\,\vec{n}_o) \in \Sigma'. \tag{A4}$$

We denote by $\delta\mathbf{q}$ the perturbation of $\mathbf{q}$. With respect to the first order we have:

$$\delta(\vec{n}_o \cdot \mathbf{q}) \approx \delta\vec{n}_o \cdot \mathbf{q} + \vec{n}_o \cdot \delta\mathbf{q}, \tag{A5}$$

$$\mathbf{q}(\vec{x} + h\,\vec{n}_o) \approx \mathbf{q}(\vec{x}) + h\,\partial_n\mathbf{q}(\vec{x}). \tag{A6}$$

Using these last relations in (A3) and (A4), and using the relations (A1) and (A2), we can obtain two new limit conditions expressed on the reference boundaries Σ (Dahlen & Tromp 1998):

$$[\delta\mathbf{q}]_-^+ = -h\,[\partial_n\mathbf{q}]_-^+ \qquad\qquad \forall\, \vec{x} \in \Sigma, \tag{A7}$$

$$[\vec{n}_o \cdot \delta\mathbf{q}]_-^+ = -h\,[\partial_n\mathbf{q} \cdot \vec{n}_o]_-^+ + \vec{\nabla}_\Sigma h \cdot [\mathbf{q}]_-^+ \quad \forall\, \vec{x} \in \Sigma, \tag{A8}$$

with $\delta\vec{n}_o = -\vec{\nabla}_\Sigma h$. These limit conditions are equivalent to (A3) and (A4) with respect to first-order perturbation.

With the two relations (A7) and (A8), we can easily express $\vec{B}_u^S$, $\vec{B}_u^F$, B_ϕ.

For B_ξ, we have to note that the perturbed variable $\delta\vec{\xi}$ can be divided into two parts:

$$\delta\vec{\xi} = \vec{\nabla}\delta\phi_1^E + 4\pi\,G(\rho_o\delta\vec{u} + \delta\rho_o\vec{u}) \tag{A9}$$

$$= \vec{\xi}(\delta\vec{u}, \delta\phi_1^E) + 4\pi\,G\delta\rho_o\vec{u}, \tag{A10}$$

then

$$\left[\vec{n}_o \cdot \vec{\xi}(\delta\vec{u}, \delta\phi_1^E)\right]_-^+ = [\vec{n}_o \cdot \delta\vec{\xi}]_-^+ - 4\pi\,G\,[\delta\rho_o\vec{u} \cdot \vec{n}_o]_-^+, \tag{A11}$$

where $[\vec{n}_o \cdot \delta\vec{\xi}]_-^+$ can be expanded following (A8) with respect to the unperturbed variable $\vec{\xi} = \vec{\xi}(\vec{u}, \phi_1^E)$.

For the traction terms $\vec{B}_{\partial\Omega}$, $\vec{B}_{\mathbf{T}}^S$ and $\vec{B}_{\mathbf{T}}^F$ the formulations are more complicated. In a planet that does not show a hydrostatic pre-stress state of equilibrium, the traction continuity does not involve the Lagrangian Cauchy stress tensor. The general continuity of traction at time t on the reference interfaces has to be expressed in terms of a tensor known as the Piola–Kirchhoff stress tensor of the first kind $\overset{=PK}{\mathbf{T}}$. The general mechanical boundary conditions for any type of earth are in reality (see Dahlen & Tromp 1998):

$$\vec{n}_o \cdot \overset{=PK}{\mathbf{T}}_1 = 0 \qquad \text{on boundary } \partial\Omega \tag{A12}$$

$$\left[\vec{n}_o \cdot \overset{=PK}{\mathbf{T}}_1\right]_-^+ = 0 \qquad \text{on boundary } \Sigma_{SS} \tag{A13}$$

$$\left[\vec{t}_1^{PK}\right]_-^+ = 0 \qquad \text{on boundary } \Sigma_{SF} \tag{A14}$$

$$\text{with} \quad \vec{t}_1^{PK} = \vec{n}_o(\vec{n}_o \cdot \vec{t}_1^{PK}) \tag{A15}$$

Where we denote by $\overset{=PK}{\mathbf{T}}_1$ the first-order perturbation of Piola–Kirchhoff tensor. This Lagrangian-Eulerian hybrid stress tensor is defined, so that a traction acting on a deformed interface of normal $\vec{n}^t$ at time t is defined: $\vec{n}^t \cdot \overset{=E}{\mathbf{T}}S^t(\vec{r}) = \vec{n}_o \cdot \overset{=PK}{\mathbf{T}}S^o(\vec{x})$ (with S^t and S^o the boundary surface at any time t and at $t = 0$). The vector $\vec{t}_1^{PK}$ is a convenient function proposed by Woodhouse & Dahlen (1978) which generalize the traction continuity condition to slipping surface boundary like Σ_{SF}. Its general expression is:

$$\vec{t}_1^{PK} = \vec{n}_o \cdot \overset{=PK}{\mathbf{T}}_1 + \vec{n}_o\vec{\nabla}_\Sigma \cdot (\varpi\vec{u}) - \varpi\vec{\nabla}_\Sigma\vec{u} \cdot \vec{n}_o, \tag{A16}$$

$$\varpi = -\vec{n}_o \cdot \overset{=}{\mathbf{T}}_o \cdot \vec{n}_o. \tag{A17}$$

One can show that the Piola–Kirchhoff tensor is linked with the Lagrangian Cauchy stress tensor with respect to first order (see Dahlen 1972; Dahlen & Tromp 1998):

$$\overset{=PK}{\mathbf{T}}_1(\vec{u}) = \overset{=L}{\mathbf{T}}_1(\vec{u}) + \overset{=}{\mathbf{T}}_o(\vec{\nabla} \cdot \vec{u}) + \vec{\nabla}\vec{u}^T \cdot \overset{=}{\mathbf{T}}_o, \tag{A18}$$

and consequently, we have the relations after some manipulations (using the continuity relations on displacement and pre-stress and that $[\vec{\nabla}_\Sigma\vec{u}]_-^+ = \vec{\nabla}_\Sigma[\vec{u}]_-^+$):

$$\vec{n}_o \cdot \overset{=L}{\mathbf{T}}_1 = \overset{=}{\mathbf{T}}_o \cdot \vec{\nabla}_\Sigma\vec{u} \cdot \vec{n}_o \qquad \text{on boundary } \partial\Omega, \tag{A19}$$

$$\left[\vec{n}_o \cdot \overset{=L}{\mathbf{T}}_1\right]_-^+ = [\overset{=}{\mathbf{T}}_o]_-^+ \cdot \vec{\nabla}_\Sigma\vec{u} \cdot \vec{n}_o \quad \text{on boundary } \Sigma_{SS}, \tag{A20}$$

$$\left[\vec{n}_o \cdot \overset{=L}{\mathbf{T}}_1\right]_-^+ = [(\overset{=}{\mathbf{T}}_o + \varpi\overset{=}{\mathbf{I}}) \cdot \vec{\nabla}_\Sigma\vec{u}]_-^+ \cdot \vec{n}_o$$
$$+ \vec{n}_o\,[\vec{u}]_-^+ \cdot \vec{\nabla}_\Sigma\varpi \quad \text{on boundary } \Sigma_{SF} \tag{A21}$$

If the pre-stress tensor is hydrostatic, we have:

$$\bar{\bar{\mathbf{T}}}_o = -p_o \bar{\bar{\mathbf{I}}} = -\varpi \bar{\bar{\mathbf{I}}}, \tag{A22}$$

$$\bar{\bar{\mathbf{T}}}_o = 0\,\bar{\bar{\mathbf{I}}} \quad \text{on boundary } \partial\Omega, \tag{A23}$$

$$[\bar{\bar{\mathbf{T}}}_o]_-^+ = 0\,\bar{\bar{\mathbf{I}}} \quad \text{on boundary } \Sigma. \tag{A24}$$

In this case, the pressure p_o is constant along the boundary discontinuities, and $\vec{\nabla}_\Sigma \varpi = \vec{0}$ on $\partial\Omega \cup \Sigma$. We see easily that the traction continuity relations (A19), (A20) and (A21) are then reduce to (10), (12) and (14) for a SNREI earth model.

We want in this work to express these traction limit conditions for a perturbed earth that presents a deviatoric pre-stress tensor with respect to a reference earth that presents a hydrostatic pre-stress equilibrium. If the tensor $\bar{\bar{\mathbf{T}}}_o$ have the properties (A23) and (A24), this is not the case for $\delta\bar{\bar{\mathbf{T}}}_o$.

Following the perturbation approach, the total perturbed vector $\bar{\bar{\mathbf{T}}}_o \cdot \vec{\nabla}_\Sigma \vec{u} \cdot \vec{n}_o + \delta(\bar{\bar{\mathbf{T}}}_o \cdot \vec{\nabla}_\Sigma \vec{u} \cdot \vec{n}_o)$ expressed on the deformed boundaries $\partial\Omega' \cup \Sigma'$ is equal to:

$$\bar{\bar{\mathbf{T}}}_o \cdot \vec{\nabla}_\Sigma \vec{u} \cdot \vec{n}_o + \delta(\bar{\bar{\mathbf{T}}}_o \cdot \vec{\nabla}_\Sigma \vec{u} \cdot \vec{n}_o) + h\,\partial_n(\bar{\bar{\mathbf{T}}}_o \cdot \vec{\nabla}_\Sigma \vec{u} \cdot \vec{n}_o) \qquad \text{on the boundaries } \partial\Omega \cup \Sigma \tag{A25}$$

with respect to first order, the last vector function become:

$$\delta\bar{\bar{\mathbf{T}}}_o \cdot \vec{\nabla}_\Sigma \vec{u} \cdot \vec{n}_o + \bar{\bar{\mathbf{T}}}_o \cdot \delta(\vec{\nabla}_\Sigma \vec{u}) \cdot \vec{n}_o + \bar{\bar{\mathbf{T}}}_o \cdot \vec{\nabla}_\Sigma \vec{u} \cdot \delta\vec{n}_o + h\,(\partial_n\bar{\bar{\mathbf{T}}}_o \cdot \vec{\nabla}_\Sigma \vec{u} \cdot \vec{n}_o + \bar{\bar{\mathbf{T}}}_o \cdot \partial_n\vec{\nabla}_\Sigma \vec{u} \cdot \vec{n}_o) \tag{A26}$$

And then using (A22), (A23) and (A24) for the reference SNREI model, and following (A8) we have:

$$\vec{n}_o \cdot \delta\bar{\bar{\mathbf{T}}}_1^L = -h\,\partial_n\bar{\bar{\mathbf{T}}}_1^L(\vec{u}) \cdot \vec{n}_o + \vec{\nabla}_\Sigma h \cdot \bar{\bar{\mathbf{T}}}_1^L(\vec{u}) + (\delta\bar{\bar{\mathbf{T}}}_o + h\,\partial_n\bar{\bar{\mathbf{T}}}_o) \cdot \vec{\nabla}_\Sigma \vec{u} \cdot \vec{n}_o \text{on boundary } \partial\Omega \tag{A27}$$

$$\left[\vec{n}_o \cdot \delta\bar{\bar{\mathbf{T}}}_1^L\right]_-^+ = -h\left[\partial_n\bar{\bar{\mathbf{T}}}_1^L(\vec{u}) \cdot \vec{n}_o\right]_-^+ + \vec{\nabla}_\Sigma h \cdot [\bar{\bar{\mathbf{T}}}_1^L(\vec{u})]_-^+ + [\delta\bar{\bar{\mathbf{T}}}_o + h\,\partial_n\bar{\bar{\mathbf{T}}}_o]_-^+ \cdot \vec{\nabla}_\Sigma \vec{u} \cdot \vec{n}_o \text{on boundary } \Sigma_{SS} \tag{A28}$$

So as for Σ_{SF}, we find:

$$\left[\vec{n}_o \cdot \delta\bar{\bar{\mathbf{T}}}_1^L\right]_-^+ = \left[-h\,\partial_n\bar{\bar{\mathbf{T}}}_1^L(\vec{u}) \cdot \vec{n}_o + \vec{\nabla}_\Sigma h \cdot \bar{\bar{\mathbf{T}}}_1^L(\vec{u}). + (\delta\bar{\bar{\mathbf{T}}}_o + \delta\varpi\bar{\bar{\mathbf{I}}}) \cdot \vec{\nabla}_\Sigma \vec{u} \cdot \vec{n}_o - \vec{n}_o\vec{u} \cdot \vec{\nabla}_\Sigma(\delta\varpi + h\,\partial_n\varpi)\right]_-^+ \tag{A29}$$

Note that $\delta\vec{\nabla}_\Sigma \varpi + h\partial_n\vec{\nabla}_\Sigma \varpi = \vec{\nabla}_\Sigma\delta\varpi + \vec{\nabla}_\Sigma(h\partial_n\varpi) - \vec{\nabla}_\Sigma(h\vec{n}_o) \cdot \vec{\nabla}_\Sigma \varpi = \vec{\nabla}_\Sigma(\delta\varpi + h\partial_n\varpi)$ because of the hydrostaticity, which impose $\vec{\nabla}_\Sigma \varpi = 0$. At last, using (20), we find the expressions for (25), (27) and (29) and the $\vec{B}_{\partial\Omega}$, $\vec{B}_{\mathbf{T}}^S$ and $\vec{B}_{\mathbf{T}}^F$.

APPENDIX B: VARIATIONAL FORM APPROXIMATION USING SPECTRAL ELEMENT METHOD

B1 Sums over the elements

The planet is divided in N volume elements Ω^e (hexahedral elements) and the discontinuity boundaries are divided in N^S surface elements Σ^e (quadrangular elements).

Each element Ω^e is constructed using a transformation $\mathcal{F}^e$ from a reference unit cubic element $\Lambda^3 = [-1, +1]^3$. We define $\mathcal{J}^e$ the Jacobian associated with $\mathcal{F}^e$. The formulation of $\mathcal{F}^e$ and $\mathcal{J}^e$ of the cubed sphere mesh are presented in Chaljub *et al.* (2003).

Let us define ξ, η and γ the Cartesian coordinate in the unit cube element $\Lambda^3 = [-1, +1]^3$. Every point $\vec{x}$ in the planet is located within an element e of the mesh and has a set of local Cartesian coordinates (ξ, η, γ) in the unit cube. We have $\vec{x} = \mathcal{F}^e(\xi, \eta, \gamma)$, and:

$$\int_\Omega q(\vec{x})\,dV = \sum_{e=1}^N \int_{\Lambda^3} q^e(\xi, \eta, \gamma)\mathcal{J}^e\,d\xi\,d\eta\,d\gamma, \tag{B1}$$

where we denote $q^e(\xi, \eta, \gamma) = q(\mathcal{F}^e(\xi, \eta, \gamma))$.

Similarly, we can approximate a sum over the Earth's surface or over the internal boundaries. We define a unit square element $\Lambda^2 = [-1, +1]^2$. In order to simplify the discretization, the surface elements have been identified with faces of the volume elements (see the left panel of Fig. 2). The surface reference element Λ^2 is chosen to correspond to a face of the volumic reference element Λ^3 where $\gamma = 1$ or $\gamma = -1$. We denote by ξ, η the Cartesian coordinates in the square unit element, they correspond to those defined in the volume element. We denote by $\mathcal{S}^{e_s}$ and $\mathcal{J}_S^{e_s}$ the transformation and its Jacobian associated to the surface element e_s. Thus we have the relation for a boundary Σ_i which contains N^s elements:

$$\int_{\Sigma_i} q(\vec{x})\,dS = \sum_{e_s=1}^{N^s} \int_{\Lambda^2} q^{e_s}(\xi, \eta)\mathcal{J}_S^{e_s}\,d\xi\,d\eta \tag{B2}$$

where we denote $q^{e_s}(\xi, \eta) = q(\mathcal{S}^{es}(\xi, \eta))$.

B2 Spectral element approximation in the unit element

The finite element method approximation is based on Lagrange polynomial interpolation of high degree. Let us consider the unit element $[-1, +1]^3$. Each interval $[-1, +1]$ is discretized. We denote by I, J and K the number of points in the three distinct dimensions of the unit element. The points are located following the Gauss–Lobatto–Legendre integration rules (each point correspond to a zero value of a Legendre polynomial define on the unit interval) (Komatitsch & Tromp 2002a). We denote by $P_i(\xi)$ the classical Lagrange polynomial in $[-1, +1]$ associated with the point number i in the interval. We can define then a polynomial interpolation of the function q:

$$q^e(\xi, \eta, \gamma) \approx \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} q^e(\xi_i, \eta_j, \gamma_k) \, P_i(\xi) P_j(\eta) P_k(\gamma). \tag{B3}$$

The Lagrangian polynomial are defined such as, on a point j of the grid: $P_i(\xi_j) = \delta_{ij}$. Using the Lagrange interpolation associated with Gauss–Lobatto–Legendre points, the discrete spatial derivative of functions can be expressed for example:

$$\frac{\partial}{\partial x} q(\vec{x}) = \left(\frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} + \frac{\partial \gamma}{\partial x} \frac{\partial}{\partial \gamma} \right) q^e(\xi, \eta, \gamma) \tag{B4}$$

$$\approx \frac{\partial \xi}{\partial x} \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} q^e(\xi_i, \eta_j, \gamma_k) \frac{\partial P_i(\xi)}{\partial \xi} P_j(\eta) P_k(\gamma) \tag{B5}$$

$$+ \frac{\partial \eta}{\partial x} \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} q^e(\xi_i, \eta_j, \gamma_k) \, P_i(\xi) \frac{\partial P_j(\eta)}{\partial \eta} P_k(\gamma) \tag{B6}$$

$$+ \frac{\partial \gamma}{\partial x} \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} q^e(\xi_i, \eta_j, \gamma_k) \, P_i(\xi) P_j(\eta) \frac{\partial P_k(\gamma)}{\partial \gamma} \tag{B7}$$

Finally, we define a sum approximation using the Gauss–Lobatto–Legendre points. In the spectral element method, the integral over $[-1, +1]$ can be approximated as:

$$\int_{\Lambda} q(\xi) \, d\xi \approx \sum_{i=1}^{I} \omega_i^I \, q(\xi_i), \tag{B8}$$

with ω_i^I the sum weight functions in the point ξ_i defined such as:

$$\omega_i^I = \int_{\Lambda} P_i(\xi) \, d\xi. \tag{B9}$$

We denote by ω^J and ω^K the weight functions associated with η and γ coordinates. In 3 dimensions, the sum approximation is:

$$\int_{\Lambda^3} q^e(\xi, \eta, \gamma) \, d\xi d\eta d\gamma \approx \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \omega_i^I \omega_j^J \omega_k^K \, q^e(\xi_i, \eta_j, \gamma_k). \tag{B10}$$

With these rules, one can construct a discrete approximation of the variational forms and extract a linear system from each equation. For example, a simple term can be approximated:

$$\int_{\Omega} \phi(\vec{x}) \, \psi(\vec{x}) \, dV = \sum_{e=1}^{N} \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \omega_i^I \omega_j^J \omega_k^K \, \phi^e(\xi_i, \eta_j, \gamma_k) \psi^e(\xi_i, \eta_j, \gamma_k), \tag{B11}$$

$$= \sum_{e=1}^{N} \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \sum_{p=1}^{I} \sum_{q=1}^{J} \sum_{r=1}^{K} \Psi_{ijk}^e \, \mathbf{M}_{IJKpqr}^e \, \Phi_{pqr}^e \tag{B12}$$

$$= \sum_{e=1}^{N} \Psi^e \cdot \mathbf{M}^e \cdot \Phi^e \tag{B13}$$

$$= \Psi \cdot \mathbf{M} \cdot \Phi \tag{B14}$$

where we denote the vectors $\Psi_{ijk}^e = \psi^e(\xi_i, \eta_j, \gamma_k)$, $\Phi_{pqr}^e = \phi^e(\xi_p, \eta_q, \gamma_r)$, and the matrix $\mathbf{M}_{IJKpqr}^e = \omega_i^I \omega_j^J \omega_k^K \delta_{Ip} \delta_{Jq} \delta_{Kr}$ which is here diagonal. $\mathbf{M}^e$ is an elementary matrix and $\mathbf{M}$ is the global matrix. We have to distinguish points defined in elements and points of the global mesh. Following Gauss–Lobatto–Legendre rules, some points of elements are located on the edges of the elements and therefore are common to multiple elements of the global volume. Before considering the global linear system (B14), the contributions from all the elements that share common global points need to be summed. This is referred in a classical finite element method to as the assembly of the system.